Chapter 7

Partial fractions

An <u>algebraic fraction</u> is a fraction in which the numerator and denominator are both polynomial expressions. A <u>polynomial expression</u> is one where every term is a multiple of a power of x, such as

$$5x^4 + 6x^3 + 7x + 4$$

The <u>degree</u> of a polynomial is the power of the highest term in x. So in this case the degree is 4.

The number in front of x in each term is called its <u>coefficient</u>. So, the coefficient of x^4 is 5. The coefficient of x^3 is 6.

Now consider the following algebraic fractions:

$$\frac{x}{x^2+2} \qquad \frac{x^3+3}{x^4+x^2+1}$$

In both cases the numerator is a polynomial of lower degree than the denominator. We call these proper fractions

With other fractions the polynomial may be of higher degree in the numerator or it may be of the same degree, for example

$$\frac{x^4+x^2+x}{x^3+x+2} \qquad \frac{x+4}{x+3}$$

- **If the degree of the numerator is less than the degree of the denominator the fraction is said to be a proper fraction**
- **If the degree of the numerator is greater than or equal to the degree of the denominator the fraction is said to be an improper fraction**

1. Revision of adding and subtracting fractions

We now revise the process for adding and subtracting fractions. Consider

$$\frac{2}{x-3} - \frac{1}{2x+1}$$

In order to add these two fractions together, we need to find the lowest common denominator. In this particular case, it is (x − 3)(2x + 1).

We write each fraction with this denominator.

$$\frac{2}{x-3} = \frac{2(2x+1)}{(x-3)(2x+1)} \quad \text{and} \quad \frac{1}{2x+1} = \frac{x-3}{(x-3)(2x+1)}$$

So

$$\frac{2}{x-3} - \frac{1}{2x+1} = \frac{2(2x+1)}{(x-3)(2x+1)} - \frac{x-3}{(x-3)(2x+1)}$$

The denominators are now the same so we can simply subtract the numerators and divide the result by the lowest common denominator to give

$$\frac{2}{x-3} - \frac{1}{2x+1} = \frac{4x+2-x+3}{(x-3)(2x+1)} = \frac{3x+5}{(x-3)(2x+1)}$$

Sometimes in mathematics we need to do this operation in reverse. In calculus, for instance, or when dealing with the binomial theorem, we sometimes need to split a fraction up into its component parts which are called partial fractions. We discuss how to do this in the following section.

2. Expressing a fraction as the sum of its partial fractions

In the previous section we saw that

$$\frac{2}{x-3}-\frac{1}{2x+1} = \frac{3x+5}{(x-3)(2x+1)}$$

Suppose we start with $\frac{3x+5}{(x-3)(2x+1)}$. How can we get this back to its component parts ?

By inspection of the denominator we see that the component parts must have denominators of x − 3 and 2x + 1 so we can write

$$\frac{3x+5}{(x-3)(2x+1)} = \frac{A}{x-3}+\frac{B}{2x+1}$$

where A and B are numbers. A and B cannot involve x or powers of x because otherwise the terms on the right would be improper fractions.

The next thing to do is to multiply both sides by the common denominator (x−3)(2x+1). This gives

$$\frac{(3x+5)(x-3)(2x+1)}{(x-3)(2x+1)} = \frac{A(x-3)(2x+1)}{x-3}+\frac{B(x-3)(2x+1)}{2x+1}$$

Then cancelling the common factors from the numerators and denominators of each term gives

$$3x + 5 = A(2x + 1) + B(x - 3)$$

Now this is an identity. This means that it is true for any values of x, and because of this we can substitute any values of x we choose into it. Observe that if we let x = −1/2 the first term on the right will become zero and hence A will disappear. If we let x = 3 the second term on the right will become zero and hence B will disappear.

If $x = -\frac{1}{2}$

$$\begin{aligned} -\frac{3}{2} + 5 &= B\left(-\frac{1}{2} - 3\right) \\ \frac{7}{2} &= -\frac{7}{2}B \end{aligned}$$

from which

$$B = -1$$

Now we want to try to find A.

If $x = 3$

$$14 = 7A$$

so that $A = 2$.
Putting these results together we have

$$\begin{aligned} \frac{3x+5}{(x-3)(2x+1)} &= \frac{A}{x-3} + \frac{B}{2x+1} \\ &= \frac{2}{x-3} - \frac{1}{2x+1} \end{aligned}$$

which is the sum that we started with, and we have now broken the fraction back into its component parts called partial fractions.

Example

Suppose we want to express $\frac{3x}{(x-1)(x+2)}$ as the sum of its partial fractions.

Observe that the factors in the denominator are $x - 1$ and $x + 2$ so we write

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

where A and B are numbers.

We multiply both sides by the common denominator (x − 1)(x + 2):

$$3x = A(x + 2) + B(x - 1)$$

This time the special values that we shall choose are x = −2 because then the first term on the right will become zero and A will disappear, and x = 1 because then the second term on the right will become zero and B will disappear.

<u>If $x = -2$</u>

$$\begin{aligned} -6 &= -3B \\ B &= \frac{-6}{-3} \\ B &= 2 \end{aligned}$$

<u>If $x = 1$</u>

$$\begin{aligned} 3 &= 3A \\ A &= 1 \end{aligned}$$

Putting these results together we have

$$\frac{3x}{(x-1)(x+2)} = \frac{1}{x-1} + \frac{2}{x+2}$$

and we have expressed the given fraction in partial fractions.

<u>Example:</u> Express the following as a sum of partial fractions

$$\frac{1}{x^3 - 9x}$$

واضح أن الكسر حقيقي، لذلك :

$$\frac{1}{x^3 - 9x} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}$$

$$\therefore 1 = A(x-3)(x+3) + Bx(x+3) + Cx(x-3)$$

بوضع $x = 0$ ينتج أن $A = -\dfrac{1}{9}$

بوضع $x = 3$ ينتج أن $B = \dfrac{1}{18}$

بوضع $x = -3$ ينتج أن $C = \dfrac{1}{18}$

$$\frac{1}{x^3 - 9x} = -\frac{1}{9x} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}.$$

Sometimes the denominator is more awkward as we shall see in the following section.

3. Fractions where the denominator has a repeated factor

Consider the following example in which the denominator has a repeated factor $(x-1)^2$.

Example

Suppose we want to express $\dfrac{3x+1}{(x-1)^2(x+2)}$ as the sum of its partial fractions.

There are actually three possibilities for a denominator in the partial fractions: $x-1$, $x+2$ and also the possibility of $(x-1)^2$, so in this case we write

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

where A, B and C are numbers.

As before we multiply both sides by the denominator (x − 1)2(x + 2) to give

$$3x + 1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \quad (1)$$

Again we look for special values to substitute into this identity. If we let x = 1 then the first and last terms on the right will be zero and A and C will disappear. If we let x = −2 the first and second terms will be zero and A and B will disappear.

If $x = 1$

$$4 = 3B \qquad \text{so that} \qquad B = \frac{4}{3}$$

If $x = -2$

$$-5 = 9C \qquad \text{so that} \qquad C = -\frac{5}{9}$$

We now need to find A. There is no special value of x that will eliminate B and C to give us A. We could use any value. We could use x = 0. This will give us an equation in A, B and C.

Since we already know B and C, this would give us A.

But here we shall demonstrate a different technique - one called equating coefficients. We take equation 1 and multiply-out the right-hand side, and then collect up like terms.

$$\begin{aligned} 3x + 1 &= A(x-1)(x+2) + B(x+2) + C(x-1)^2 \\ &= A(x^2 + x - 2) + B(x+2) + C(x^2 - 2x + 1) \\ &= (A + C)x^2 + (A + B - 2C)x + (-2A + 2B + C) \end{aligned}$$

This is an identity which is true for all values of x. On the left-hand side there are no terms involving x^2 whereas on the right we have $(A + C)x^2$. The only way this can be true is if

$$A + C = 0$$

This is called **equating coefficients** of x^2. We already know that $C = -\frac{5}{9}$ so this means that $A = \frac{5}{9}$. We also already know that $B = \frac{4}{3}$. Putting these results together we have

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{5}{9(x-1)} + \frac{4}{3(x-1)^2} - \frac{5}{9(x+2)}$$

and the problem is solved.

<u>Example:</u> Express the following as a sum of partial fractions

$$\frac{2x^2+x+2}{(1+x)(1-x)^2}$$

<u>**الحل**</u>

$$\frac{2x^2+x+2}{(1+x)(1-x)^2}=\frac{A}{1+x}+\frac{B}{1-x}+\frac{C}{(1-x)^2}$$

$$\therefore 2x^2+x+2=A(1-x)^2+B(1-x)(1+x)+C(1+x)$$

بوضع $x = -1$ في الطرفين ينتج أن

$$2=4A \Rightarrow A=\frac{1}{2}$$

بوضع $x = 1$ في الطرفين ينتج أن

$$5=2C \Rightarrow C=\frac{5}{2}$$

بمساواة معامل x^2 في الطرفين ينتج أن

$$2=A-B \Rightarrow B=A-2=\frac{1}{2}-2=-\frac{3}{2}$$

$$\therefore \frac{2x^2+x+2}{(1+x)(1-x)^2}=\frac{1}{2(1+x)}-\frac{3}{2(1-x)}+\frac{5}{2(1-x)^2}\,.$$

4. Fractions in which the denominator has a quadratic term

Sometimes we come across fractions in which the denominator has a quadratic term which cannot be factorised. We will now learn how to deal with cases like this.

Example: **Suppose we want to express**

$$\frac{5x}{(x^2+x+1)(x-2)}$$

as the sum of its partial fractions.

Note that the two denominators of the partial fractions will be (x^2+x+1) and ($x-2$). When the denominator contains a quadratic factor we have to consider the possibility that the numerator can contain a term in x. This is because if it did, the numerator would still be of lower degree than the denominator - this would still be a proper fraction. So we write

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

As before we multiply both sides by the denominator ($x^2 + x + 1$)($x - 2$) to give

$$5x = (Ax + B)(x - 2) + C(x^2 + x + 1)$$

One special value we could use is x = 2 because this will make the first term on the right-hand side zero and so A and B will disappear.

If $x = 2$

$$10 = 7C \qquad \text{and so} \qquad C = \frac{10}{7}$$

Unfortunately there is no value we can substitute which will enable us to get rid of C so instead we use the technique of equating coefficients. We have

$$\begin{aligned} 5x &= (Ax+B)(x-2) + C(x^2+x+1) \\ &= Ax^2 - 2Ax + Bx - 2B + Cx^2 + Cx + C \\ &= (A+C)x^2 + (-2A+B+C)x + (-2B+C) \end{aligned}$$

We still need to find A and B. There is no term involving x^2 on the left and so we can state that

$$A + C = 0$$

Since $C = \frac{10}{7}$ we have $A = -\frac{10}{7}$.

The left-hand side has no constant term and so

$$-2B + C = 0 \qquad \text{so that} \qquad B = \frac{C}{2}$$

But since $C = \frac{10}{7}$ then $B = \frac{5}{7}$. Putting all these results together we have

$$\begin{aligned} \frac{5x}{(x^2+x+1)(x-2)} &= \frac{-\frac{10}{7}x + \frac{5}{7}}{x^2+x+1} + \frac{\frac{10}{7}}{x-2} \\ &= \frac{-10x+5}{7(x^2+x+1)} + \frac{10}{7(x-2)} \\ &= \frac{5(-2x+1)}{7(x^2+x+1)} + \frac{10}{7(x-2)} \end{aligned}$$

Example: **Express the following as a sum of partial fractions**

$$\frac{x-1}{(x+1)(x^2+2x+2)}$$

الحل

واضح أن أحد عوامل المقام من الدرجة الثانية ولا يمكن تحليله، لذلك :

$$\frac{x-1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$$

$$\therefore x-1 = A(x^2+2x+2)+(Bx+C)(x+1)$$

بوضع x=-1 في الطرفين ينتج أن $A=-2$

بمساواة معامل x^2 **في الطرفين ينتج أن**

$$0 = A+B \Rightarrow B=-A=2$$

.

بوضع x=0 في الطرفين (أي بمساواة الحد المطلق في الطرفين) ينتج أن

$$-1 = 2A+C \Rightarrow C=-2A-1=4-1=3$$

$$\therefore \frac{x-1}{(x+1)(x^2+2x+2)} = \frac{-2}{x+1}+\frac{2x+3}{x^2+2x+2}$$

<u>**Example:**</u> **Express the following as a sum of partial fractions**

$$\frac{1}{x^4+x^2+1} \ .$$

<u>**الحل**</u>

$$x^4+x^2+1 = x^4+2x^2+1-x^2 = (x^2+1)^2-x^2 = (x^2+x+1)(x^2-x+1)$$

$$\frac{1}{x^4+x^2+1} = \frac{A_1x+B_1}{x^2+x+1}+\frac{A_2x+B_2}{x^2-x+1}$$

$$\therefore 1 = (A_1x+B_1)(x^2-x+1)+(A_2x+B_2)(x^2+x+1)$$

بمقارنة معامل x^3 ، x^2 ، x ، الحد المطلق في الطرفين نحصل على المعادلات التالية :

$$A_1+A_2=0\ ,\ B_1-A_1+A_2+B_2=0,\ \ B_1+B_2=1\ ,\ A_1-B_1+A_2+B_2=0$$

بحل هذه المعادلات نحصل على

$$A_1=\frac{1}{2},\ A_2=-\frac{1}{2},\ B_1=B_2=\frac{1}{2}$$

$$\therefore \frac{1}{x^4+x^2+1} = \frac{x+1}{2(x^2+x+1)}+\frac{-x+1}{2(x^2-x+1)}$$

Exercises

Express the following as a sum of partial fractions

a) $\dfrac{2x-1}{(x+2)(x-3)}$ b) $\dfrac{2x+5}{(x-2)(x+1)}$ c) $\dfrac{3}{(x-1)(2x-1)}$ d) $\dfrac{1}{(x+4)(x-2)}$

Express the following as a sum of partial fractions

a) $\dfrac{5x^2+17x+15}{(x+2)^2(x+1)}$ b) $\dfrac{x}{(x-3)^2(2x+1)}$ c) $\dfrac{x^2+1}{(x-1)^2(x+1)}$

Express the following as a sum of partial fractions

a) $\dfrac{x^2-3x-7}{(x^2+x+2)(2x-1)}$ b) $\dfrac{13}{(2x+3)(x^2+1)}$ c) $\dfrac{x}{(x^2-x+1)(3x-2)}$

Express the following as a sum of powers of x and partial fractions

(1) $\dfrac{2x+5}{(x+2)(x+3)}$ (2) $\dfrac{x^2+20}{(x-2)^2(x+4)}$ (3) $\dfrac{2+x}{1-x^2}$

(4) $\dfrac{x^3}{(x+4)(x-1)}$ (5) $\dfrac{8x-1}{(x-2)(x^2+1)}$ (6) $\dfrac{3x^2+x+9}{(x+3)(x^2+x+5)}$

(7) $\dfrac{x-3-2x^2}{x^2(x-1)}$ (8) $\dfrac{x^2+3x-13}{(x+2)(x-1)}$ (9) $\dfrac{3x^3+6x^2+17x+1}{(x+3)(x^2+4)}$

(10) $\dfrac{2x^2+11}{(x^2+4)(x-3)}$ (11) $\dfrac{x^4-3x^3-3}{x^2(x-1)}$ (12) $\dfrac{x^3+2x^2+61}{(x+3)^2(x^2+4)}$

Differential equations - Introduction

A **differential equation** is an equation involving a variable and its derivatives with respect to one or more independent variables. Differential equations often arise in modelling real world phenomena — derivatives give rates of change, and rates of change are often empirically measurable.

The **order** of the equation is the order of the highest-order derivative that it contains. If there is a single independent variable, the equation is an **ordinary differential equation (ODE)**; if there are several independent variables, it is a **partial differential equation (PDE)**.

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 e^x \qquad \text{first order, ordinary}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{second order, partial}$$

To **solve** the differential equation means (roughly) to find an expression for the dependent variable in terms of the independent variables which satisfies the original equation.

Example. $\displaystyle\frac{dy}{dx} = 2(1 - y^2)$

The solution is

$$y = \frac{ce^x - 1}{ce^x + 1}$$

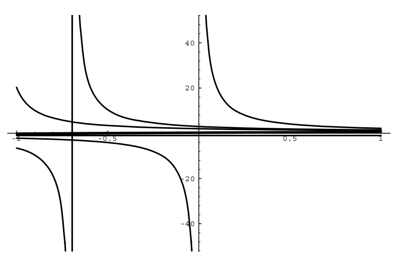
c is an **arbitrary constant**. That is, the expression above is a solution for any value of c: $c = 1$, $c = \pi$, $c = -7.9$, and so on.

You can verify that $y = \dfrac{ce^x - 1}{ce^x + 1}$ solves the equation by plugging it into both sides and checking that the equation is true:

$$\frac{dy}{dx} = \frac{2ce^x}{(ce^x + 1)^2}, \qquad \frac{1}{2}(1 - y^2) = \frac{2ce^x}{(ce^x + 1)^2}.$$

It is good to remember that you can check the solution to a differential equation by plugging in.

Note that each value of c gives a different solution $y = \dfrac{ce^x - 1}{ce^x + 1}$. Intuitively, the original equation involves a first derivative. You "undo" a *first* derivative by integrating *once*. A *single* integration produces *one* arbitrary constant.



The picture shows the solution curves for $c = -3, -2, -1, 0, 1, 2, 3$. The solution curves for different values of c form a family of curves which fill up the plane. They may remind you of the **integral curves** of a vector field. Indeed, the two situations are closely related. $\square$

Take a first order equation

$$\frac{dy}{dx} = f(x, y).$$

$\dfrac{dy}{dx}$ is the slope of a solution curve, so the equation says that $f(x, y)$ is the slope of a solution curve at the point (x, y). For example, suppose

$$\frac{dy}{dx} = \frac{x}{y+1}.$$

The slope of the solution curve passing through the point $(4, 1)$ is $\dfrac{dy}{dx} = \dfrac{4}{1+1} = 2$.

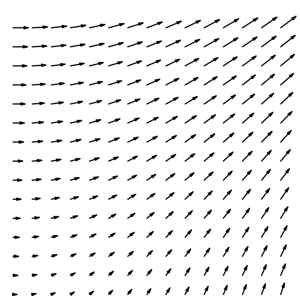
It follows that you can get a rough picture of the solution curves by drawing a little segment at each point (x, y) such that the slope of the segment is $f(x, y)$. You could do this by hand with a piece of graph paper; you can also use q computer equipped with the appropriate software. The symbolic math package *Mathematica* has a function called `PlotVectorField` which draws a picture of a vector field. Here's how to use it.

First, you will need to load the package containing the function:

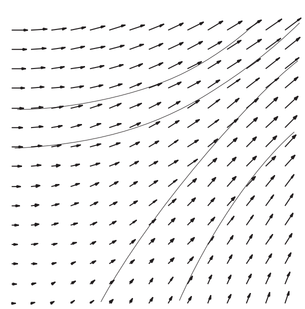
```
Needs["Graphics‘PlotField‘"]
```

I'll use $\dfrac{dy}{dx} = \dfrac{x}{y+1}$ as an example. Think of the fraction as dy divided by dx, with $dy = x$ and $dx = y + 1$. The vector field is $\langle y + 1, x\rangle$. The following command draws a picture of the field:

```
PlotVectorField[{y + 1, x}, {x, 0, 3}, {y, 0, 3}]
```

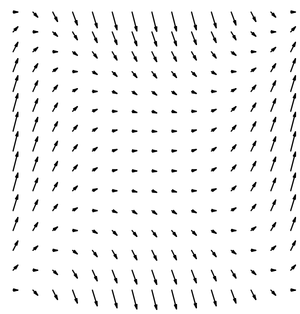


You can get the solution curves by sketching in curves which follow the arrows:



What do you do with something like $\dfrac{dy}{dx} = x^2 - y^2$? It isn't obviously a fraction. Just choose dx and dy so the quotient is $x^2 - y^2$. For example, $dx = 1$ and $dy = x^2 - y^2$ will work:

```
PlotVectorField[{1, x^2 - y^2}, {x, -2, 2}, {y, -2, 2}]
```



The pictures above are called **direction fields**. Note that you can draw them *without* actually solving the equation. Hence, you can sometimes tell things about the solution curves without actually solving the equation.

Generically, the **general solution** to an n-th order differential equation has n arbitrary constants. To put things informally, the general solution is an expression which contains all possible solutions as special cases.

This course is primiarly concerned with **ordinary differential equations**. **Partial differential equations** are often more difficult to solve, and may require techniques such as **Fourier series**.

Example. Verify that $u = x^2 + t^2$ is a solution to

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

(This equation is a special case of the **wave equation**.)

$$u_{xx} = 2 = u_{tt}. \quad \square$$

Example. Find the values of r such that $y = e^{rx}$ is a solution to

$$y'' - 2y' - 3y = 0.$$

(The derivatives are taken with respect to x.)

Compute the first and second derivatives:

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

Plug them into the differential equation and solve for r:

$$y'' - 2y' - 3y = r^2 e^{rx} - 2re^{rx} - 3e^{rx} = (r^2 - 2r - 3)e^{rx} = 0.$$

Then $r^2 - 2r - 3 = 0$, or $(r - 3)(r + 1) = 0$, so $r = 3$ or $r = -1$.

e^{3x} and e^{-x} are solutions to the equation. □

Remarks.

1. The previous example shows that if you can guess the *form* of a solution to a differential equation, you can often obtain a solution.

2. An equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

is a **linear equation** in y. The dependent variable y and its derivatives only occur to the first power, with coefficients which are functions of x alone. □

First-order linear equations

An equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

is a **linear equation** in y. The dependent variable y and its derivatives only occur to the first power, with coefficients which are functions of x alone.

Here is a **first-order linear** equation:

$$a(x)y' + b(x)y = c(x).$$

Divide through by $a(x)$:

$$y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}.$$

Rename the fractions:

$$y' + P(x)y = Q(x).$$

You should write first-order linear equations in this standard form before using the solution method below.

The idea for solving this equation is to try to turn the left side into an **exact** form — i.e. something which is exactly $\dfrac{df}{dx}$ for some f. To do this, multiply both sides by the **integrating factor**

$$I = \exp \int P(x)\,dx = e^{\int P(x)\,dx}.$$

Here is why it works. By the Product Rule and the Chain Rule,

$$\frac{d}{dx}\left(y \exp \int P(x)\,dx\right) = y'e^{\int P(x)\,dx} + y\frac{d}{dx}\left(e^{\int P(x)\,dx}\right) =$$

$$y'e^{\int P(x)\,dx} + ye^{\int P(x)\,dx} \cdot \frac{d}{dx}\left(\int P(x)\,dx\right) = y'e^{\int P(x)\,dx} + ye^{\int P(x)\,dx} \cdot P(x).$$

The last expression is just $e^{\int P(x)\,dx}$ times the left side of our original differential equation. So multiply the original equation by $e^{\int P(x)\,dx}$:

$$y'e^{\int P(x)\,dx} + yPe^{\int P(x)\,dx} = Qe^{\int P(x)\,dx} = IQ.$$

As above, the left side is the derivative of $ye^{\int P(x)\,dx}$, so

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = IQ,$$

$$ye^{\int P(x)\,dx} = \int IQ\,dx,$$

$$yI = \int IQ\,dx.$$

In doing a problem, you can simply compute $I = e^{\int P(x)\,dx}$, then jump to the last equation. To finish, compute the integral on the right side.

Example. $\dfrac{dy}{dx} + \dfrac{3}{x} y = \dfrac{\cos x}{x^2}$.

First, compute the integrating factor:

$$I = \exp \int \frac{3}{x}\, dx = \exp 3 \ln x = \exp \ln x^3 = x^3.$$

(This cancellation of exp and ln often occurs in these computations. Note that you have to push the constant into the exponent first.)

Now plug the integrating factor into the equation $yI = \int IQ\, dx$:

$$yx^3 = \int x^3 \frac{\cos x}{x^2}\, dx = \int x \cos x\, dx.$$

Compute the integral on the right using integration by parts:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & \cos x \\ & & \searrow & \\ - & 1 & & \sin x \\ & & \searrow & \\ + & 0 & \rightarrow & -\cos x \end{array}$$

$$\int x \cos x\, dx = x \sin x + \cos x + C$$

Hence,

$$yx^3 = x \sin x + \cos x + C, \quad y = \frac{\sin x}{x^2} + \frac{\cos x}{x^3} + \frac{C}{x^3}. \quad \square$$

Example. $y' - \dfrac{\sin x}{\cos x} y = (\sin x)^5$, $y(0) = 1$.

The "$y(0) = 1$" is called an **initial condition**. This means you are to find the solution which satisfies $x = 0$, $y = 1$ — i.e. the solution which passes through the point $(0, 1)$. To do this, plug $x = 0$ and $y = 1$ into the general solution and solve for the arbitrary constant.

The integrating factor is

$$I = \exp\left(-\int \frac{\sin x}{\cos x}\, dx\right) = \exp \ln \cos x = \cos x.$$

Therefore,

$$y \cos x = \int (\sin x)^5 \cos x\, dx = \frac{1}{6} (\sin x)^6 + C,$$

$$y = \frac{1}{6} \frac{(\sin x)^6}{\cos x} + \frac{C}{\cos x}.$$

Now plug in the initial condition:

$$1 = \frac{1}{6} \cdot 0 + C, \quad \text{so} \quad C = 1.$$

The solution is

$$y = \frac{1}{6}\frac{(\sin x)^6}{\cos x} + \frac{1}{\cos x}. \quad \square$$

Example. $y\,dx + (3x - xy + 2)\,dy = 0$.

This equation is not linear in y:

$$\frac{dy}{dx} + \frac{y}{3x - xy + 2} = 0.$$

However, it is linear in x:

$$\frac{dx}{dy} + \frac{3}{y}x - x + \frac{2}{y} = 0, \quad \frac{dx}{dy} + \left(\frac{3}{y} - 1\right)x = -\frac{2}{y}.$$

The integrating factor is

$$I = \exp\int\left(\frac{3}{y} - 1\right)\,dy = y^3 e^{-y}.$$

Therefore,

$$xy^3e^{-y} = -2\int y^2e^{-y}\,dy = -2\left(-y^2e^{-y} - 2ye^{-y} - 2e^{-y}\right) + C = 2y^2e^{-y} + 4ye^{-y} + 4e^{-y} + C.$$

The solution is

$$x = \frac{2}{y} + \frac{4}{y^2} + \frac{4}{y^3} + \frac{C}{y^3}e^y.$$

Here's the work for the integral:

$$\begin{array}{cccc}
 & \dfrac{d}{dy} & & \int dy \\
+ & y^2 & & e^{-y} \\
 & & \searrow & \\
- & 2y & & -e^{-y} \\
 & & \searrow & \\
+ & 2 & & e^{-y} \\
 & & \searrow & e^{-y} \\
- & 0 & \rightarrow & -e^{-y}
\end{array}$$

$$\int y^2e^{-y}\,dy = -y^2e^{-y} - 2ye^{-y} - 2e^{-y} + C. \quad \square$$

Example. $y' = 2y + e^{2x}\cos 3x$, $y(0) = 4$.

Rewrite the equation as $y' - 2y = e^{2x}\cos 3x$.
The integrating factor is

$$I = \exp\int -2\,dx = e^{-2x}.$$

(*A standard mistake here is to use* 2 *instead of* -2. But the form I used in setting things up was $y' + P(x)y = Q(x)$, with a "+" on the left. So if the y term is subtracted, the "$-$" is used in computing I.)
Therefore,

$$ye^{-2x} = \int e^{-2x}e^{2x}\cos 3x\,dx = \int \cos 3x\,dx = \frac{1}{3}\sin 3x + C.$$

The general solution is

$$y = \frac{1}{3}e^{2x}\sin 3x + Ce^{2x}.$$

Plug in the initial condition:

$$4 = y(0) = 0 + C, \quad C = 4.$$

The solution is

$$y = \frac{1}{3}e^{2x}\sin 3x + 4e^{2x}. \quad \square$$

Example. (Discontinuous forcing) $y' + \dfrac{3}{x}y = g(x)$, where

$$g(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ \dfrac{1}{x} & \text{if } x > 1 \end{cases}, \quad \text{and} \quad y\left(\frac{1}{2}\right) = \frac{1}{8}.$$

The idea is to solve the equation separately on $0 \le x \le 1$ and on $x > 1$, then match the pieces up at $x = 1$ to get a *continuous* solution.

$0 \le x \le 1$: $y' + \dfrac{3}{x}y = 1$. The integrating factor is

$$I = \exp\int \frac{3}{x}\,dx = e^{3\ln x} = x^3.$$

Then

$$yx^3 = \int x^3\,dx = \frac{1}{4}x^4 + C.$$

The solution is

$$y = \frac{1}{4}x + \frac{C}{x^3}.$$

Plug in the initial condition:

$$\frac{1}{8} = y\left(\frac{1}{2}\right) = \frac{1}{8} + 8C, \quad C = 0.$$

The solution on the interval $0 \le x \le 1$ is

$$y = \frac{1}{4}x.$$

Note that $y(1) = \dfrac{1}{4}$.

$x > 1$: $y' + \dfrac{3}{x}y = \dfrac{1}{x}$. The integrating factor is the same as before, so

$$yx^3 = \int x^2\,dx = \frac{1}{3}x^3 + C.$$

The solution is

$$y = \frac{1}{3} + \frac{C}{x^3}.$$

In order to get this piece to "match" with the previous piece, set $y(1) = \dfrac{1}{4}$:

$$\frac{1}{4} = y(1) = \frac{1}{3} + C, \quad C = -\frac{1}{12}.$$

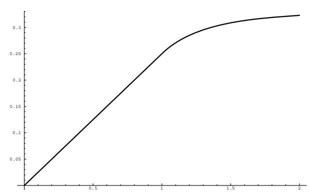
The solution on the interval $x > 1$ is

$$y = \frac{1}{3} - \frac{1}{12}\frac{1}{x^3}.$$

The complete solution is

$$y = \begin{cases} \dfrac{1}{4}x & \text{if } 0 \le x \le 1 \\ \dfrac{1}{3} - \dfrac{1}{12}\dfrac{1}{x^3} & \text{if } x > 1 \end{cases}$$

You can see the two pieces glued together in the picture below:



□

Example. Calvin Butterball's backpack has a capacity of 5 gallons. Calvin's creative lab partners pour 1 gallon of pure water into the backpack. After that, water containing 0.5 pounds of dissolved salt per gallon is pumped in at 2 gallons per minute; the well-stirred mixture drains out the bottom at 1 gallon per minute. How many pounds of salt are dissolved in the solution in the backpack at the instant when it overflows?

Let S be the amount of salt (in pounds) dissolved in the solution in the backpack at time t. Write down the **rate equation** for S; it is the inflow rate minus the outlow rate:

$$\frac{dS}{dt} = \left(0.5\frac{\text{lbs}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{S \text{ lbs}}{1+t \text{ gal}}\right)\left(1\frac{\text{gal}}{\text{min}}\right).$$

In both terms, I've multiplied the concentration by the flow rate. Everything is straightforward except perhaps the $\dfrac{S}{1+t}$ term. This is the concentration of salt in the tank at time t. Why? First, S *is* the amount of salt (in pounds) dissolved in the solution. Now there is 1 gallon in the backpack initially, and the volume increases by $2 - 1 = 1$ gallon each minute — so after t minutes, there are $1 + t$ gallons. Thus, the concentration is $\dfrac{S}{1+t}$. This goes into the outflow term, because it's the concentration of salt in the fluid draining out.

Notice that $\dfrac{dS}{dt}$ has the units pounds per minute. And if you cancel the gallon units on the right side, everything on the right has the units pounds per minute as well. This serves as a check that you've written down something sensible.

Rearrange the equation:

$$\frac{dS}{dt} = 1 - \frac{S}{1+t}, \quad \frac{dS}{dt} + \frac{S}{1+t} = 1.$$

Find the integrating factor:

$$I = \exp\int \frac{1}{1+t}\,dt = 1 + t.$$

Therefore,

$$S(1+t) = \int (1+t)\,dt = \frac{1}{2}(1+t)^2 + C.$$

When $t = 0$, $S = 0$ (because there was pure water in the backpack initially):

$$0 \cdot 1 = \frac{1}{2} + C, \quad C = -\frac{1}{2}.$$

Now

$$S(1+t) = \frac{1}{2}(1+t)^2 - \frac{1}{2},$$

$$S = \frac{1}{2}(1+t) - \frac{1}{2}\frac{1}{1+t}.$$

Finally, when does the backpack overlow? The capacity is 5 gallons, there is 1 gallon initially, and the volume increases by 1 gallon each minute. Hence, it overlows when $t = 4$:

$$S = \frac{1}{2}(1+4) - \frac{1}{2}\frac{1}{1+4} = 2.4 \text{ pounds.} \quad \square$$

It is important to know when a differential equation has a solution (the **existence problem**). Some equations have solutions only for certain sets of initial conditions; I'll give an example below of an equation with *no solutions.*

It's also important to know, if a solution is found, whether it is the only possible solution. Geometrically, the question is whether there is a single solution curve passing through a given point. This is called the **uniqueness problem**. I will give an example later on of an equation with infinitely many solution curves passing through a point.

These questions can be answered for certain classes of differential equations.

Consider the initial value problem

$$y' + P(x)y = Q(x), \qquad y(x_0) = y_0.$$

The **existence and uniqueness theorem** for first-order linear equations says that if P and Q are continuous on an interval (a, b), then there is a *unique* solution satisfying the initial condition.

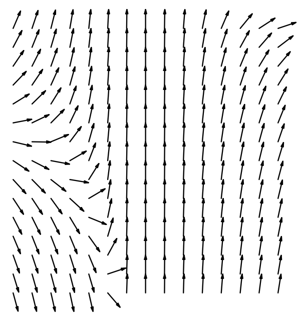
Example. $\dfrac{dy}{dx} + \dfrac{3}{x-1}y = 6\dfrac{x+1}{(x-1)^2}$, $y(2) = 9$.

$P = \dfrac{3}{x-1}$; it is continuous for $x > 1$ and for $x < 1$. $Q = 6\dfrac{x+1}{(x-1)^2}$; it is continuous for $x > 1$ and for $x < 1$. The initial condition is $x = 2$, $y = 9$. Since $x = 2$ lies in the interval $x > 1$, there is a unique solution to the initial value problem for $x > 1$. *Notice that you know this without solving the equation!*

In fact, the solution is

$$y = \frac{2x^3 - 6x + 5}{(x-1)^3}, \qquad x > 1.$$

Here is the direction field for the equation:



Notice the singularity along the line $x = 1$. $\square$

Example. $xy' + 2y = 3x$ has only one solution defined at $x = 0$.

To see this, rewrite the equation as $y' + \dfrac{2}{x}y = 3$.

The integrating factor is

$$I = \exp \int \frac{2}{x}\, dx = x^2.$$

Then

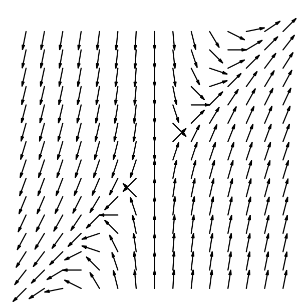
$$yx^2 = \int 3x^2\, dx = x^3 + C.$$

The solution is

$$y = x + \frac{C}{x^2}.$$

$y = x$ is a solution (set $C = 0$), and it's defined at $x = 0$. However, if $C \neq 0$, the solution is not defined at $x = 0$.

Here's the direction field:



Notice that the solution $y = x$ is the only solution that crosses the singularity at $x = 0$.

On the other hand, the solutions change dramatically if the equation is changed just a little. Consider $xy' - 2y = 3x$. Rewrite it as $y' - \dfrac{2}{x}y = 3$.

The integrating factor is

$$I = \exp \int -\frac{2}{x}\, dx = x^{-2}.$$

Then

$$yx^{-2} = \int \frac{3}{x^2}\,dx = -\frac{3}{x} + C.$$

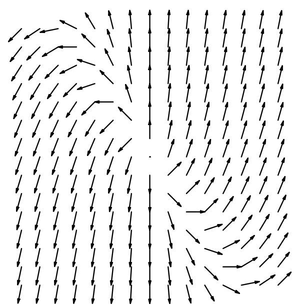
The solution is

$$y = -3x + Cx^2.$$

In this case, y is defined for all x, for all values of C.

However, note that the initial value problem $y(0) = 0$ has infinitely many solutions, since $y(0) = 0$ for any value of C. On the other hand, if $y_0 \neq 0$, the initial value problem $y(0) = y_0$ has no solutions.

Here's the direction field:



You can see all the solutions curves emanating from the origin, corresponding to the infinitely many solutions to the initial value problem with $y(0) = 0$. □

The existence and uniqueness theorem stated above applies to first-order linear equations. There are similar results for other classes of equations. However, the situation for an arbitrary differential equation is often not so nice.

Example. (An equation with exactly one solution) Suppose that

$$\left(\frac{dy}{dx}\right)^2 + y^2 = 0 \quad \text{for} \quad a < x < b.$$

A sum of squares is 0 if and only if both terms are 0. Therefore, $y = 0$, and this is the only possible solution. □

Example. (An equation with no solutions) Suppose that

$$\left(\frac{dy}{dx}\right)^2 + x^2 = 0$$

has a solution $y = f(x)$ on the interval $a < x < b$. Then $x = 0$ for $a < x < b$, which is ridiculous. ⨯Hence, the equation has no solution on any open interval. □

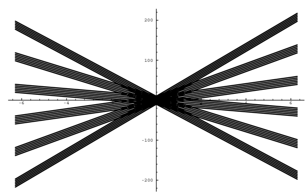
Example. $\cos y' = 0$.

$\cos \text{junk} = 0$ when junk is an odd multiple of $\frac{\pi}{2}$. Thus,

$$y' = (2n+1)\frac{\pi}{2}, \quad \text{so} \quad y = (2n+1)\frac{\pi}{2}x + C, \qquad n \in \mathbf{Z}.$$

This is actually a two-parameter family of solutions. That is $y = \dfrac{5\pi}{2}x + C$ is a family of solutions, $y = \dfrac{\pi}{2}x + C$ is a family of solutions, and so on. In this situation, there are infinitely many solutions passing through each point.

I've drawn the family of solutions curves below for n and c each going from -10 to 10 in increments of 4.



You can see that it looks as though many curves pass through a given point. $\square$

Separation of variables

In some cases, you can solve a differential equation

$$f\left(x, y, y'\right) = 0$$

by moving all the x's to one side and the y's to the other. Then solve the equation by integrating both sides. This is called **separation of variables**.

Example. $x^2\,dx + y(x-1)\,dy = 0$.

Separate:

$$x^2\,dx + y(x-1)\,dy = 0$$

$$-\int \frac{x^2}{x-1}\,dx = \int y\,dy$$

Integrate:

$$-\int \left(x + 1 + \frac{1}{x-1}\right)\,dx = \int y\,dy$$

$$-\left(\frac{1}{2}x^2 + x + \ln|x-1|\right) + C = \frac{1}{2}y^2$$

$$-x^2 - x - 2\ln|x-1| + C_0 = y^2$$

Observe that there is one *integration step*, hence only one constant.

Note also that in the last line I replaced $2C$ with C_0. It would not be wrong to write $2C$, but this is neater. You can always rename constant quantities to make the result look nicer.

Finally, the problem did not include an initial condition; hence, I've stopped at y^2, rather than taking square roots. Without an initial condition, I can't tell which square root to take. $\square$

Example. **(Exponential growth or decay)** Let a be a constant. The **exponential growth or decay equation** describes a situation in which a variable grows or shrinks at a rate proportional to the amount present:

$$\frac{dy}{dx} = ay.$$

Separate:

$$\frac{dy}{dx} = ay, \quad \int \frac{dy}{y} = \int a\,dx.$$

Integrate and solve for y:

$$\ln|y| = ax + C, \quad |y| = e^{ax+C} = e^C e^{ax}, \quad y = C_0 e^{ax}.$$

(I've replaced $\pm e^C$ with C_0.) If $a > 0$, then y increases as x increases: exponential *growth*. If $a < 0$, then y decreases as x decreases: exponential *decay*. $\square$

Example. **(Logistic growth)** In the real world, things cannot grow without bound. In many cases, there is a natural limit to the ability of an environment to support the growth of a population. For example, there are always limits to the food supply and space.

In many cases, this situation is modelled by the **logistic equation**. Let a be a constant. The logistic equation is

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right).$$

Separate:

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right)$$

$$\int \frac{dN}{N\left(1 - \frac{N}{K}\right)} = \int a\,dt$$

$$K\int \frac{dN}{N(K-N)} = \int a\,dt$$

Compute the integral on the left by partial fractions:

$$\frac{1}{N(K-N)} = \frac{A}{N} + \frac{B}{K-N}$$

$$1 = A(K-N) + BN$$

Set $N = 0$; then $1 = KA$, so $A = \dfrac{1}{K}$. Set $N = K$; $1 = KB$, so $B = \dfrac{1}{K}$. Therefore,

$$\frac{1}{N(K-N)} = \frac{1}{K}\left(\frac{1}{N} + \frac{1}{K-N}\right).$$

Back to the integration:

$$\int \left(\frac{1}{N} + \frac{1}{K-N}\right) dN = \int a\,dt$$

$$\ln|N| - \ln|K-N| = at + C$$

Now solve for N in terms of t:

$$\ln\left|\frac{N}{K-N}\right| = at + C$$

$$\left|\frac{N}{K-N}\right| = e^{at+C} = e^C e^{at}$$

$$\frac{N}{K-N} = C_0 e^{at}$$

$$N = KC_0e^{at} - C_0e^{at}N$$

$$N\left(1 + C_0e^{at}\right) = KC_0e^{at}$$

$$N = \frac{KC_0e^{at}}{1 + C_0e^{at}}$$

Note that $\lim_{t\to\infty} N = K$. Thus, K is the limiting population. It is often called the **carrying capacity**, the largest population that the environment can support. $\square$

Example. (Dropping solutions) Consider the equation

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}.$$

Separate:

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$

$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3)\,dx$$

Integrate and solve for y:

$$3(y+1)^{1/3} = \frac{1}{2}(x-3)^2 + C$$

$$(y+1)^{1/3} = \frac{1}{6}(x-3)^2 + C_0$$

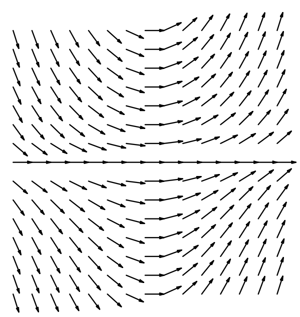
$$y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3$$

$$y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$$

All of this looks routine. However, note that $y = -1$ is a solution to the *original* equation:

$$\frac{dy}{dx} = 0 \quad \text{and} \quad (x-3)(y+1)^{2/3} = 0.$$

You can see the solution $y = -1$ as a horizontal line in the direction field below:



However, you can't obtain $y = -1$ from $y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$ by setting the constant C_0 equal to a number. (You'd need to find a constant which makes $\frac{1}{6}(x-3)^2 + C_0 = 0$ *for all* x.)

Two points emerge from this.

1. You can often drop solutions by performing certain algebraic operations (in this case, division).
2. You don't always get every solution to a differential equation by assigning values to the arbitrary constants. $\square$

Example. **(Equations of the form $y' = f(ax+by+c)$)** A standard rule of thumb is to substitute for an expression which appears "a lot" in an equation or expression. A differential equation

$$y' = f(ax+by+c)$$

can be reduced to a separable equation by the substitution $v = ax+by+c$.

Consider the equation $y' = (x + y + 1)^2$. Let $v = x + y + 1$, so $v' = 1 + y'$. Then

$$v' - 1 = v^2$$

$$\frac{dv}{dx} = v^2 + 1$$

$$\int \frac{dv}{v^2 + 1} = \int dx$$

$$\arctan v = x + C$$

$$v = \tan(x + C)$$

$$x + y + 1 = \tan(x + C)$$

$$y = \tan(x + C) - x - 1. \quad \square$$

9-28-1998

Exact Equations and Integrating Factors

An equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is **exact** if

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N \quad \text{for some } f(x,y).$$

This is the same as saying that the vector field $\langle M(x,y), N(x,y)\rangle$ is a **gradient field** (or a **conservative field**) — in fact, $\langle M(x,y), N(x,y)\rangle = \nabla f$.

The reason this is important is that an exact equation can be integrated. Here's an example:

$$(3x^2y - 3y)\,dx + (x^3 - 3x)\,dy = 0$$

If $f(x,y) = x^3y - 3xy$, then

$$\frac{\partial f}{\partial x} = 3x^2y - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = x^3 - 3x.$$

Therefore, the equation may be rewritten as

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0, \quad \text{or} \quad df = 0.$$

Integrating both sides gives $f = C$, i.e. $x^3y - 3xy = C$. The differential equation is solved.

It's useful, then, to be able to tell when an equation $M\,dx + N\,dy = 0$ is exact. This amounts to determining if $\langle M, N\rangle$ is a conservative field. This is a problem in multivariable calculus, and the solution is well known: With reasonable conditions on M and N, the field $\langle M, N\rangle$ is conservative if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Example. Solve $(\sin y - \sin x)\,dx + (x\cos y + 1)\,dy = 0$, $y(0) = 1$.

$$M = \sin y - \sin x, \qquad N = x\cos y + 1, \quad \text{so} \quad \frac{\partial M}{\partial y} = \cos y, \qquad \frac{\partial N}{\partial x} = \cos y.$$

The equation is exact.

I need to find a function f such that

$$\frac{\partial f}{\partial x} = \sin y - \sin x \quad \text{and} \quad \frac{\partial f}{\partial y} = x\cos y + 1.$$

I can use the partial integration technique which is used to recover a potential function for a conservative field.

Integrate $M = \dfrac{\partial f}{\partial x}$ with respect to x:

$$\frac{\partial f}{\partial x} = \sin y - \sin x, \quad \text{so} \quad f = \int (\sin y - \sin x)\,dx = x\sin y + \cos x + g(y).$$

Here g is constant with respect to x, so it is a function of y alone.

Now differentiate with respect to y:

$$x\cos y + \frac{dg}{dy} = \frac{\partial f}{\partial y} = x\cos y + 1.$$

This means that
$$\frac{dg}{dy} = 1 \quad \text{so} \quad g(y) = y + h.$$

h is a numerical constant, which I may take to be 0. Then
$$f = x\sin y + \cos x + y.$$

The original equation becomes $df = 0$, so $f = C$ by integrating both sides. The solution is
$$x\sin y + \cos x + y = C.$$

(A common mistake is to write $f = x\sin y + \cos x + y$ for the solution. However, this is just the potential function. The solution to a first-order equation ought to contain an arbitrary constant — hence, $x\sin y + \cos x + y = C$.)

Finally, plug in the initial condition $x = 0$, $y = 1$:
$$0 \cdot \sin 1 + \cos 0 + 1 = C, \quad C = 2.$$

The solution is
$$x\sin y + \cos x + y = 2. \quad \square$$

Example. Solve $\dfrac{dy}{dx} = \dfrac{\dfrac{1}{y} - 2xy^4 - 4x}{\dfrac{x}{y^2} + 4x^2y^3}$.

The equation is not separable, nor is it first-order linear in x or in y. Rewrite the equation as
$$\left(\frac{1}{y} - 2xy^4 - 4x\right) dx - \left(\frac{x}{y^2} + 4x^2y^3\right) dy = 0.$$

Now
$$M = \frac{1}{y} - 2xy^4 - 4x, \qquad N = -\frac{x}{y^2} - 4x^2y^3, \quad \text{so} \quad \frac{\partial M}{\partial y} = -\frac{1}{y^2} - 8xy^3, \qquad \frac{\partial N}{\partial x} = -\frac{1}{y^2} - 8xy^3.$$

The equation is exact. I must find an f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$.

Integrate $M = \dfrac{\partial f}{\partial x}$ with respect to x:
$$\frac{\partial f}{\partial x} = \frac{1}{y} - 2xy^4 - 4x, \quad \text{so} \quad f = \int \left(\frac{1}{y} - 2xy^4 - 4x\right) dx = \frac{x}{y} - x^2y^4 - 2x^2 + g(y).$$

Now differentiate with respect to y:
$$-\frac{x}{y^2} - 4x^2y^3 + \frac{dg}{dy} = \frac{\partial f}{\partial y} = -\frac{x}{y^2} - 4x^2y^3.$$

Therefore,
$$\frac{dg}{dy} = 0 \quad \text{and} \quad g(y) = h = 0.$$

Hence,
$$f = \frac{x}{y} - x^2y^4 - 2x^2 = C. \quad \square$$

Example. The equation

$$\left(\frac{6y}{x} - 6y^2\right)\,dx + (3 - 4xy)\,dy = 0$$

is not exact, because

$$\frac{\partial N}{\partial x} = -4y \quad \text{while} \quad \frac{\partial M}{\partial y} = \frac{6}{x} - 12y.$$

In some cases (such as this one), *it may be possible to multiply by something which will make the equation exact.* Suppose that something is called P. I want this equation to be exact:

$$PM\,dx + PN\,dy = 0.$$

This means that

$$\frac{\partial PN}{\partial x} = \frac{\partial PM}{\partial y}.$$

In general, you can't solve this for P without some other conditions. Suppose that P *is a function of* x *only.* Then

$$P\frac{\partial N}{\partial x} + N\frac{dP}{dx} = P\frac{\partial M}{\partial y}.$$

Then

$$\frac{\partial P}{\partial x} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}P.$$

If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x (but not y), this equation is separable. I can solve it for P in terms of x. Then I multiply the original equation by P to get an exact equation, and I solve the resulting exact equation.

Going back to the example,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{\dfrac{6 - 8xy}{x}}{3 - 4xy} = \frac{2}{x}.$$

By the derivation above, the integrating factor P satisfies

$$\frac{\partial P}{\partial x} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}P \quad \text{or} \quad \frac{dP}{dx} = \frac{2}{x}P.$$

Separating variables and integrating yields $P = x^2$. Now go back and multiply the original equation by x^2; it becomes

$$(6xy - 6x^2y^2)\,dx + (3x^2 - 4x^3y)\,dy = 0.$$

This equation is exact:

$$\frac{\partial M}{\partial y} = 6x - 12x^2y, \quad \frac{\partial N}{\partial x} = 6x - 12x^2y.$$

You can check for yourself that the solution is

$$3x^2y - 2x^3y^2 = C. \quad \square$$

There is a similar result which applies when $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only. I'll summarize these two results below.

Given an equation $M\,dx + N\,dy = 0$ which is not exact:

1. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x alone, then an integrating factor P is given by

$$P = \exp \int \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}\,dx.$$

2. If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y alone, then an integrating factor P is given by

$$P = \exp \int \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}\,dy.$$

Find the integrating factor, multiply the original equation by the integrating factor, then solve the resulting exact equation.

As a matter of strategy, then, if $\dfrac{\partial N}{\partial x} \neq \dfrac{\partial M}{\partial y}$, find the difference $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ and divide it by M (respectively by N) to see if you get a function of y alone (respectively x alone). Note that you use $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}$ in the x case but $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ in the y case: the sign *does* make a difference!

It is also possible to find integrating factors in other (more complicated) cases.

Homogeneous equations

A function $f(x, y)$ is **homogeneous of degree** n in x and y if

$$f(ax, ay) = a^n f(x, y).$$

Roughly, this means that the "total power" of x and y is the same in all the terms of $f(x, y)$. Here are some examples.

Example. $\sin \dfrac{x}{y}$ is homogeneous of degree 0:

$$\sin \frac{ax}{ay} = \sin \frac{x}{y} = a^0 \sin \frac{x}{y}. \quad \square$$

Example. $\dfrac{2x - 3y}{5x + 4y}$ is also homogeneous of degree 0:

$$\frac{2ax - 3ay}{5ax + 4ay} = \frac{2x - 3y}{5x + 4y} = a^0 \frac{2x - 3y}{5x + 4y}. \quad \square$$

Example. $\cos x$ is not homogeneous of any degree:

$$\cos ax \neq a^n \cos x$$

is not an identity for any n. $\square$

Example. $4x^5 - 7x^3y^2 + xy^4$ is homogeneous of degree 5:

$$4(ax)^5 - 7(ax)^3(ay)^2 + (ax)(ay)^4 = a^5 \left(4x_7^5x^3y^2 + xy^4\right). \quad \square$$

Here is how this applies to differential equations. A first-order equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is **homogeneous** if M and N are homogeneous functions *of the same degree.*

Example. The equation

$$(x^2 - 3xy)\, dx + (7x^2 - y^2)\, dy = 0$$

is homogeneous, since $x^2 - 3xy$ and $7x^2 - y^2$ are homogeneous of degree 2.

On the other hand,

$$(x + 5y)\, dx - (x^2 + 4y^2)\, dy = 0$$

is not homogeneous; $x + 5y$ and $x^2 + 4y^2$ are individually homogeneous, but not of the same degree.

$$(\sin x - \cos y)\,dx + x\cos y\,dy = 0$$

is not homogeneous, since $\sin x - \cos y$ and $x\cos y$ are not homogeneous. $\square$

The following two facts can be used to simplify a homogeneous differential equation.

Fact 1: If M and N are homogeneous of the same degree, then $\dfrac{M}{N}$ is homogeneous of degree 0.

Proof:

$$\frac{M(ax, ay)}{N(ax, ay)} = \frac{a^n M(x, y)}{a^n N(x, y)} = \frac{M(x, y)}{N(x, y)} = a^0 \frac{M(x, y)}{N(x, y)}. \quad \square$$

Fact 2: If f is homogeneous of degree 0, then f can be expressed as a function of $\dfrac{y}{x}$.

Proof: Since f is homogeneous of degree 0, $f(ax, ay) = a^0 f(x, y) = f(x, y)$ is an identity. Set $a = \dfrac{1}{x}$:

$$f\left(1, \frac{y}{x}\right) = f(x, y).$$

The left side is a function of $\dfrac{y}{x}$. $\square$

Now suppose

$$M\,dx + N\,dy = 0$$

is homogeneous. Rewrite it as

$$\frac{dy}{dx} = -\frac{M}{N}.$$

The right side is homogeneous of degree 0 (Fact 1), so it can be written as a function of $\dfrac{y}{x}$ (Fact 2). Suppose then that

$$-\frac{M}{N} = g\left(\frac{y}{x}\right).$$

Let $y = vx$, so $\dfrac{y}{x} = v$. Then

$$-\frac{M}{N} = g(v),$$

and by the Product Rule,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

The original equation becomes

$$v + x\frac{dv}{dx} = g(v) \quad \text{or} \quad \frac{dv}{dx} = \frac{g(v) - v}{x}.$$

This equation can be solved by separation of variables.

Example. Solve $y' = \dfrac{3x - y}{x + y}$.

The right side is clearly homogeneous of degree 0.

Let $y = vx$, so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substitute:

$$v + x\frac{dv}{dx} = \frac{3-v}{1+v}, \quad x\frac{dv}{dx} = \frac{3-v}{1+v} - v = \frac{(3+v)(1-v)}{1+v}.$$

Separate:

$$\int \frac{1+v}{(3+v)(1-v)}\,dv = \int \frac{dx}{x}.$$

Decompose the integrand on the left using partial fractions:

$$\frac{1+v}{(3+v)(1-v)} = \frac{A}{3+v} + \frac{B}{1-v}$$

$$1 + v = A(1-v) + B(3+v)$$

Setting $x = 1$ yields $2 = 4B$, so $B = \dfrac{1}{2}$. Setting $x = -3$ yields $-2 = 4A$, so $A = -\dfrac{1}{2}$. Therefore,

$$\frac{1+v}{(3+v)(1-v)} = \frac{1}{2}\left(-\frac{1}{3+v} + \frac{1}{1-v}\right).$$

Now

$$\int \frac{1}{2}\left(-\frac{1}{3+v} + \frac{1}{1-v}\right)dv = \int \frac{dx}{x}, \quad \frac{1}{2}\left(-\ln|3+v| - \ln|1-v|\right) = \ln|x| + C.$$

Combine the logs on the left, then exponentiate to kill the logs:

$$\ln|(3+v)(1-v)| = -2\ln|x| - 2C, \quad (3+v)(1-v) = \frac{C_0}{x^2}.$$

Finally, put y back:

$$\left(3 + \frac{y}{x}\right)\left(1 - \frac{y}{x}\right) = \frac{C_0}{x^2}, \quad (3x+y)(x-y) = C_0. \quad \square$$

Example. Solve $(x - y\ln y + y\ln x)\,dx + x(\ln y - \ln x)\,dy = 0$.

Rewrite the equation as

$$\left(x - y\ln\frac{y}{x}\right)\,dx + x\ln\frac{y}{x}\,dy = 0.$$

$x - y\ln\dfrac{y}{x}$ and $x\ln\dfrac{y}{x}$ are homogeneous of degree 1.
Rearrange the equation:

$$\frac{dy}{dx} = \frac{y\ln\dfrac{y}{x} - x}{x\ln\dfrac{y}{x}}.$$

The right side is homogeneous of degree 0. Let $y = vx$, so $v = \dfrac{y}{x}$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substitute:

$$v + x\frac{dv}{dx} = \frac{xv\ln v - x}{x\ln v} = \frac{v\ln v - 1}{\ln v}, \quad x\frac{dv}{dx} = \frac{v\ln v - 1}{\ln v} - v = \frac{1}{\ln v}.$$

Separate:

$$\int \ln v\,dv = \int \frac{1}{x}\,dx.$$

Integrate $\ln v$ by parts:

$$\begin{array}{cccc} & \dfrac{d}{dv} & & \int dv \\ + & \ln v & & 1 \\ & & \searrow & \\ - & \dfrac{1}{v} & \rightarrow & v \end{array}$$

Therefore,

$$\int \ln v \, dv = v \ln v - \int \frac{1}{v} \cdot v \, dv = v \ln v - \int dv = v \ln v - v + C.$$

Hence,

$$v \ln v - v = \ln x + C.$$

Put y back:

$$\frac{y}{x} \ln \frac{y}{x} - \frac{y}{x} = \ln x + C, \quad y \ln \frac{y}{x} - y = x \ln x + Cx. \quad \square$$

Example. Solve $(x + y + 1) \, dx + (x + 2y - 3) \, dy = 0$.

This would be homogeneous if the "1" and "3" weren't there. The idea is to make a preliminary substitution

$$x = u + h, \quad y = v + k.$$

I will *choose* h and k so that the result is homogeneous.
Since $dx = du$ and $dy = dv$,

$$(u + v + h + k + 1) \, du + (u + 2v + h + 2k - 3) \, dv = 0.$$

I want to pick h and k so that the constant terms go away:

$$h + k + 1 = 0, \quad h + 2k - 3 = 0.$$

Solving simultaneously, I obtain $k = 4$, $h = -5$. The substitution is

$$x = u - 5, \quad y = v + 4.$$

With this substitution, the equation becomes

$$(u + v) \, du + (u + 2v) \, dv = 0, \quad \text{or} \quad \frac{dv}{du} = -\frac{u + v}{u + 2v}.$$

Let $v = wu$, so $w = \dfrac{v}{u}$ and $\dfrac{dv}{du} = w + u \dfrac{dw}{du}$.
Then

$$w + u \frac{dw}{du} = -\frac{u + wu}{u + 2wu} = -\frac{1 + w}{1 + 2w}, \quad u \frac{dw}{du} = -\frac{1 + w}{1 + 2w} - w = \frac{2w^2 + 2w + 1}{2w + 1}.$$

Separate:

$$\int \frac{2w + 1}{2w^2 + 2w + 1} \, dw = -\int \frac{du}{u}, \quad \frac{1}{2} \ln |2w^2 + 2w + 1| = -\ln |u| + C.$$

Put v bacK;

$$\frac{1}{2} \ln \left| 2 \left(\frac{v}{u} \right)^2 + 2 \frac{v}{u} + 1 \right| = -\ln |u| + C.$$

Put x and y back:

$$\frac{1}{2}\ln\left|2\left(\frac{y-4}{x+5}\right)^2 + 2\frac{y-4}{x+5} + 1\right| = -\ln|x+5| + C. \quad \square$$

Example. Solve $(x+y+1)\,dx + (2x+2y-1)\,dy = 0$.

This looks like the previous problem. But if you let

$$x = u + h, \quad y = v + k,$$

and then try to choose h and k so the constant terms go away, you'll get stuck!
Reason: The h and k equations become

$$h + k = -1, \quad 2h + 2k = 1,$$

and these equations are inconsistent — there are no solutions.
Instead, let $z = x + y$, so $dz = dx + dy$. Substitute and eliminate x:

$$(z+1)(dz - dy) + (2z-1)\,dy = 0, \quad 1 - \frac{dz}{dy} = \frac{2z-1}{z+1}, \quad -\frac{dz}{dy} = \frac{2z-1}{z+1} - 1 = \frac{z-2}{z+1}.$$

Separate:

$$-\int \frac{z+1}{z-2}\,dz = \int dy, \quad -z - 3\ln|z-2| = y + C.$$

Put x back:

$$-x - y - 3\ln|x+y-2| = y + C. \quad \square$$

A Review of Elementary Solution Methods

Here is a list of the kinds of equations I've discussed so far:

1. Separable equations.
2. Exact equations.
3. Homogeneous equations.
4. First-order linear equations.
5. Bernoulli and Riccati equations.
6. Equations requiring clever substitutions.
7. Linear constant coefficient homogeneous equations.

Linear constant coefficient homogeneous equations are straightforward, and I won't review them here. There are two things involved in solving the other types of equations:

1. You need to know *how* to apply each technique.
2. You need to know *which* technique to apply in a given problem.

Sometimes, it is simply a matter of trying one technique after another. However, this doesn't mean that you should use the first thing that works — there may be an easier way. Take the time to think about how each of the methods would work in a given problem.

Example. $(2x - 3y + 1)\,dx - (3x + 2y - 4)\,dy = 0$.

Write the equation as

$$(2x - 3y + 1)\,dx = (3x + 2y - 4)\,dy.$$

Evidently, there is no way to separate the x's and y's.

The equation is not homogeneous; however, it could be converted into a homogeneous equation by the substitutions $x = u + a$, $y = v + b$. After making the substitutions, you'd need to solve for a and b so as to make the constant terms vanish.

This method will work, though it is a little tedious.

Even when you have a method that will work, it is often wise to look at the problem a little longer to see if there is an easier way.

The equation does not seem to be first-order linear. On the other hand,

$$\frac{\partial M}{\partial y} = -3 \quad \text{and} \quad \frac{\partial N}{\partial x} = -3,$$

so the equation is exact. The method of exact equations is usually easier than the method of homogeneous equations, so I'll use exact equations rather than the substitution I noticed earlier.

I must find an f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$. Integrate M with respect to x:

$$f = \int (2x - 3y + 1)\,dx = x^2 - 3xy + x + g(y).$$

Compute $\dfrac{\partial f}{\partial y}$ and set it equal to N:

$$-(3x + 2y - 4) = \frac{\partial f}{\partial y} = -3x + \frac{dg}{dy}.$$

Therefore,

$$\frac{dg}{dy} = -2y + 4, \quad g = -y^2 + 4y.$$

Therefore, $f = x^2 - 3xy + x - y^2 + 4y$. The solution is

$$x^2 - 3xy + x - y^2 + 4y = C. \quad \square$$

Example. $xy' = y + \sqrt{y^2 - x^2}$.

The equation is not separable. Solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{y^2 - x^2}}{x}.$$

It is not first-order linear in y.
Solve for $\dfrac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{x}{y + \sqrt{y^2 - x^2}}.$$

It is not first-order linear in x.
Check for exactness. Write the equation as

$$(y + \sqrt{y^2 - x^2})\,dx - x\,dy = 0.$$

Then

$$\frac{\partial M}{\partial y} = 1 + \frac{y}{\sqrt{y^2 - x^2}} \quad \text{and} \quad \frac{\partial N}{\partial x} = -1.$$

It is not exact.
It better be homogeneous!

$$\frac{dy}{dx} = \frac{y + \sqrt{y^2 - x^2}}{x} = \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 - 1}.$$

The right side is a function of $\dfrac{y}{x}$; the equation is homogeneous.
Let $y = vx$, so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Then

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{v^2x^2 + x^2}}{x} = v + \sqrt{v^2 + 1}, \quad x\frac{dv}{dx} = \sqrt{v^2 + 1}.$$

Separate variables:

$$\int \frac{dv}{\sqrt{v^2 + 1}} = \int \frac{1}{x}\,dx, \quad \ln\left|\sqrt{v^2 + 1} + v\right| = \ln|x| + C.$$

I'll do the v-integral separately:

$$\int \frac{dv}{\sqrt{v^2 + 1}} = \int \frac{(\sec\theta)^2}{\sqrt{(\tan\theta)^2 + 1}}\,d\theta =$$

$$\int \frac{(\sec\theta)^2}{\sqrt{(\sec\theta)^2}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|\sqrt{v^2+1} + v| + C.$$

Put y back:

$$\ln\left|\sqrt{\left(\frac{y}{x}\right)^2 + 1} + \frac{y}{x}\right| = \ln|x| + C.$$

Exponentiate both sides and rename the constant:

$$\sqrt{\left(\frac{y}{x}\right)^2 + 1} + \frac{y}{x} = C_0 x. \quad \square$$

Example. $(3x^2y^3 - 2y)\,dx - x\,dy = 0$.

The equation is clearly not homogeneous or separable.

$$\frac{\partial M}{\partial y} = 9x^2y^2 - 2 \quad \text{but} \quad \frac{\partial N}{\partial x} = -1.$$

It is not exact.
Solve for $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$:

$$\frac{dy}{dx} = 3xy^3 - \frac{2}{x}y \quad \text{and} \quad \frac{dx}{dy} = \frac{x}{3x^2y^3 - 2y}.$$

It is not first-order linear in x or y.
Rearrange the $\dfrac{dy}{dx}$ equation:

$$\frac{dy}{dx} + \frac{2}{x}y = 3xy^3.$$

It is a Bernoulli equation. Let $v = y^{1-3} = y^{-2}$. Then $\dfrac{dv}{dx} = -y^{-3}\dfrac{dy}{dx}$. Multiply the equation by $-2y^{-3}$:

$$-2y^{-3}\frac{dy}{dx} - \frac{4}{x}y^{-2} = -6x.$$

Substitute:

$$\frac{dv}{dx} - \frac{4}{x}v = -6x.$$

This is first order linear in v. The integrating factor is

$$I = \exp\int -\frac{4}{x}\,dx = x^{-4}.$$

Therefore,

$$vx^{-4} = \int -6x^{-3}\,dx = 3x^{-2} + C, \quad v = 3x^2 + Cx^4.$$

Put the y's back:

$$y^{-2} = 3x^2 + Cx^4, \quad y^2 = \frac{1}{3x^2 + Cx^4}. \quad \square$$

Example. $\dfrac{dy}{dx} = \tan y \cot x - \sec y \cos x$.

The equation is not first-order linear in either variable. It is not separable, nor is it homogeneous. It is not Bernoulli.

Is it exact? Rearrange it:

$$(\sin x - \sin y)\cos x\, dx + \sin x \cos y\, dy = 0.$$

Therefore,

$$\frac{\partial M}{\partial y} = -\cos y \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos x \cos y.$$

It is not exact!

The idea here is to try to substitute to simplify the equation. The test of whether a substitution is the right one is whether it works! One rule of thumb is to look for common expressions — expressions that appear in several places. Another rule of thumb is to look for substitutions that eliminate one variable or another. In this vein, it is good to look for u-du combinations.

In the equation $(\sin x - \sin y)\cos x\, dx + \sin x \cos y\, dy = 0$ notice the "$\cos y\, dy$" at the end, the differential of $\sin y$. Try $u = \sin y$, so $du = \cos y\, dy$:

$$(\sin x - u)\cos x\, dx + \sin x\, du = 0, \qquad \frac{du}{dx} - u\frac{\cos x}{\sin x} = -\cos x.$$

The equation is first-order linear in u.

The integrating factor is

$$I = \exp \int -\frac{\cos x}{\sin x}\, dx = \exp - \ln(\sin x) = \frac{1}{\sin x}.$$

Hence,

$$u\frac{1}{\sin x} = -\int \frac{\cos x}{\sin x}\, dx = -\ln|\sin x| + C.$$

Solve for u:

$$u = -\sin x \ln|\sin x| + C \sin x.$$

Put y back:

$$\sin y = -\sin x \ln|\sin x| + C \sin x. \quad \square$$

Example. A tank contains 20 gallons of pure water. Water containing 2 pounds of dissolved yogurt per gallon enters the tank at 4 gallons per minute. The well-stirred mixture drains out at 4 gallons per minute. How much yogurt is dissolved in the tank mixutre after 10 minutes? Find the limiting amount of yogurt in the tank as $t \to \infty$.

Let Y be the number of pounds of dissolved yogurt at time t.

$$\frac{dY}{dt} = \text{inflow} - \text{outflow} = \left(4\frac{\text{gal}}{\text{min}}\right)\left(2\frac{\text{lb}}{\text{gal}}\right) - \left(4\frac{\text{gal}}{\text{min}}\right)\left(\frac{Y\text{ lb}}{20\text{ gal}}\right).$$

Then

$$\frac{dY}{dt} + \frac{Y}{5} = 8.$$

You can do this by separation or by using an integrating factor. I will do the latter:

$$I = \exp \int \frac{1}{5}\, dt = \exp \frac{t}{5}.$$

Then

$$Y \exp \frac{t}{5} = \int \exp \frac{t}{5}\, dt = 40 \exp \frac{t}{5} + C.$$

The solution is

$$Y = 40 + C \exp \frac{-t}{5}.$$

Initially, there is no yogurt in the tank:

$$0 = Y(0) = 40 + C, \quad \text{so} \quad C = -40.$$

Therefore,

$$Y = 40 - 40 \exp \frac{-t}{5}.$$

When $t = 10$,

$$Y(10) = 40 - 40e^{-2} \approx 34.58659.$$

As $t \to \infty$, $\exp \dfrac{-t}{5} \to 0$, so $Y \to 40$. In the limit, the amount of dissolved yogurt approaches 40 pounds. This makes sense, since the tank is being flushed with water containing 2 pounds of yogurt per gallon, and the tank holds 20 gallons. $\square$

Chapter 3

Infinite sequences and series

Infinite sequences

<u>Definition:</u>

An infinite sequence of numbers is a function whose domain is the set of positive integers.

♦ A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

the first term a_1, the second term a_2, and so on the nth term a_n.

♦ The integer n is called the index of a_n.

♦ We can think of the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

as a function that sends 1 to a_1, 2 to a_2 and in general sends the positive integer n to nth term a_n.

♦ The sequence can be written as $\{a_n\}$.

♦ The sequence

$$\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots$$

can be written

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$

or

$$\{a_n\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}.$$

♦ The sequence $1, 2, 3, 4, \ldots$ is not the same as the sequence $2, 1, 3, 4, \ldots$.

Convergence and divergence

Definition:

The sequence $\{a_n\}$ converges to the number L if for all $\varepsilon > 0$ there exists an integer N such that for all n

$$n > N \Rightarrow |a_n - L| < \varepsilon .$$

If no such number L exists, we say that $\{a_n\}$ diverges.

♦ If $\{a_n\}$ converges to L, we write

$$\lim_{n \to \infty} a_n = L$$

or simply $a_n \to L$, and call L the limit of the sequence $\{a_n\}$.

Remark:

If $x > 0$, then there exists an integer N such that

$$x > \frac{1}{N} .$$

Example (1):

By using the definition, prove that

(1) $\lim_{n \to \infty} \frac{1}{n} = 0$ (2) $\lim_{n \to \infty} \frac{n-1}{n} = 1$ (3) $\lim_{n \to \infty} \frac{1}{2^n} = 0$.

Solution:

(1) $\lim_{n \to \infty} \frac{1}{n} = 0$

Let $\varepsilon > 0$ be given. Now we must show that there exists an integer N such that for all n

$$n > N \quad \Rightarrow |\frac{1}{n} - 0| < \varepsilon \;\Rightarrow |\frac{1}{n}| < \varepsilon \quad \Rightarrow \frac{1}{n} < \varepsilon$$

$\because \varepsilon > 0$, from the above remark, there exists an integer N such that

$$\varepsilon > \frac{1}{N}. \tag{1}$$

If $n > N \quad \Rightarrow \quad \frac{1}{n} < \frac{1}{N}$ (2)

then from (1) and (2) we get

$$\frac{1}{n} < \varepsilon.$$

Then $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

(2) $\lim\limits_{n\to\infty} \frac{n-1}{n} = 1$

Let $\varepsilon > 0$ be given. Now we must show that there exists an integer N such that for all n

$$n > N \quad \Rightarrow |\frac{n-1}{n} - 1| < \varepsilon \;\Rightarrow |\frac{n-1-n}{n}| < \varepsilon \;\Rightarrow |\frac{-1}{n}| < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon$$

$\because \varepsilon > 0$, there exists an integer N such that

$$\varepsilon > \frac{1}{N}. \tag{3}$$

If $n > N \quad \Rightarrow \quad \frac{1}{n} < \frac{1}{N}$ (4)

then from (3) and (4) we get

$$\frac{1}{n} < \varepsilon.$$

Then $\lim\limits_{n\to\infty} \dfrac{n-1}{n} = 1$.

(3) $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$

Let $\varepsilon > 0$ be given. Now we must show that there exists an integer N such that for all n

$$n > N \quad \Rightarrow \; \left| \frac{1}{2^n} - 0 \right| < \varepsilon \;\; \Rightarrow \; \left| \frac{1}{2^n} \right| < \varepsilon \;\; \Rightarrow \; \frac{1}{2^n} < \varepsilon$$

$\because 2^n \geq n$ for $n \in \mathbb{N} \quad \Rightarrow \dfrac{1}{2^n} \leq \dfrac{1}{n}$ (5)

If $n > N \quad \Rightarrow \dfrac{1}{n} < \dfrac{1}{N}$ (6)

$\because \varepsilon > 0$, there exists an integer N such that

$$\varepsilon > \frac{1}{N}. \tag{7}$$

then from (5), (6) and (7) we get

$$\frac{1}{2^n} < \varepsilon .$$

Then $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$.

Definition: (diverges to infinity)

The sequence $\{a_n\}$ diverges to infinity if for every number M there exists an integer N such that for all

$$n > N \;\Rightarrow\; a_n > M \;.$$

If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \text{ or } a_n \to \infty .$$

Similarly if for every number m there exists an integer N such that for all

$$n > N \Rightarrow a_n < m,$$

then

$$\lim_{n\to\infty} a_n = -\infty \text{ or } a_n \to -\infty.$$

Calculating limits of sequences

Theorem (1):

If $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$ and k is a constant, then

(1) $\lim_{n\to\infty} k = k$

(2) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

(3) $\lim_{n\to\infty} k\, a_n = k \lim_{n\to\infty} a_n$

(4) $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

(5) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$; $B \neq 0$.

Example (2):

By using Theorem 1, find the following limit:

(1) $\lim_{n\to\infty} \left(-\dfrac{1}{n} \right) = -\lim_{n\to\infty} \dfrac{1}{n} = -1 \cdot 0 = 0$.

(2) $\lim_{n\to\infty} \left(\dfrac{n-1}{n} \right) = \lim_{n\to\infty} \left(\dfrac{\frac{n-1}{n}}{\frac{n}{n}} \right) = \lim_{n\to\infty} \left(\dfrac{1-\frac{1}{n}}{1} \right) = \dfrac{\lim_{n\to\infty} \left(1 - \frac{1}{n} \right)}{\lim_{n\to\infty} 1} = \dfrac{\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 1} = \dfrac{1-0}{1} = 1$

(3) $\lim_{n\to\infty} \left(\dfrac{5}{n^2} \right) = 5 \lim_{n\to\infty} \dfrac{1}{n} \cdot \lim_{n\to\infty} \dfrac{1}{n} = 5 \cdot 0 \cdot 0 = 0$.

(4) $\lim\limits_{n\to\infty}\left(\dfrac{4-7n^6}{n^6+3}\right)=\lim\limits_{n\to\infty}\left(\dfrac{\frac{4-7n^6}{n^6}}{\frac{n^6+3}{n^6}}\right)=\lim\limits_{n\to\infty}\left(\dfrac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right)=\dfrac{\lim\limits_{n\to\infty}\left(\frac{4}{n^6}-7\right)}{\lim\limits_{n\to\infty}\left(1+\frac{3}{n^6}\right)}=\dfrac{0-7}{1+0}=-7$.

Theorem (2) (The Sandwich Theorem for sequences)

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L .$$

Example (3):

Prove that (by using Theorem 2)

(1) $\lim\limits_{n\to\infty}\dfrac{\cos n}{n}=0$ (2) $\lim\limits_{n\to\infty}\dfrac{1}{2^n}=0$ (3) $\lim\limits_{n\to\infty}(-1)^n\dfrac{1}{n}=0$.

Solution:

(1) $\lim\limits_{n\to\infty}\dfrac{\cos n}{n}=0$

$\because |\cos n|\le 1 \;\Rightarrow\; -1\le \cos n \le 1 \;\Rightarrow\; -\dfrac{1}{n}\le \dfrac{\cos n}{n}\le \dfrac{1}{n}$

$\because \lim\limits_{n\to\infty}\dfrac{1}{n}=0$, then from Sandwich Theorem $\lim\limits_{n\to\infty}\dfrac{\cos n}{n}=0$.

(2) $\lim\limits_{n\to\infty}\dfrac{1}{2^n}=0$

$\because 2^n \ge n \;\Rightarrow\; 0\le \dfrac{1}{2^n}\le \dfrac{1}{n}$.

$\because \lim\limits_{n\to\infty}\dfrac{1}{n}=0$, then from Sandwich Theorem $\lim\limits_{n\to\infty}\dfrac{1}{2^n}=0$.

(3) $\lim\limits_{n\to\infty}(-1)^n\dfrac{1}{n}=0$

$\because\quad -\dfrac{1}{n}\le(-1)^n\dfrac{1}{n}\le\dfrac{1}{n}$

$\because \lim\limits_{n\to\infty}\dfrac{1}{n}=0$, then from Sandwich Theorem $\lim\limits_{n\to\infty}(-1)^n\dfrac{1}{n}=0$.

Theorem (3): (The Continuous Function Theorem for Sequences)
Let $\{a_n\}$ be a sequence of real numbers. If $a_n\to L$ and if f is a function that is continuous at L and defined at all a_n then $f(a_n)\to f(L)$.

Example (4):

Show that, applying Theorem 3,

(1) $\sqrt{\dfrac{n+1}{n}}\to 1$ (2) $2^{1/n}\to 1$.

Solution:

(1) $\sqrt{\dfrac{n+1}{n}}\to 1$

Taking $a_n=\dfrac{n+1}{n}$ and $f(x)=\sqrt{x}$.

$$\because \lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1}=1 \Rightarrow L=1.$$

Then, by Theorem 3,

$$f(a_n)\to f(L) \Rightarrow \sqrt{\frac{n+1}{n}}\to\sqrt{1}=1.$$

(2) $2^{1/n}\to 1$

Taking $a_n=\dfrac{1}{n}$ and $f(x)=2^x$.

$$\because \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow L = 0.$$

Then, by Theorem 3,

$$f(a_n) \to f(L) \Rightarrow 2^{1/n} \to 2^0 = 1.$$

Theorem (4):

The following six sequences converge to the limits listed below:

1- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2- $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3- $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4- $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6- $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In the formulas (3) and (6), x remains fixed as $n \to \infty$.

Example (5):

By using Theorem 4, find the following limits

(1) $\lim_{n\to\infty} \frac{\ln n^2}{n}$ (2) $\lim_{n\to\infty} \sqrt[n]{n^2}$ (3) $\lim_{n\to\infty} \sqrt[n]{3n}$

(4) $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n$ (5) $\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n$ (6) $\lim_{n\to\infty} \frac{100^n}{n!}$

Solution:

(1) $\lim_{n\to\infty} \frac{\ln n^2}{n} = \lim_{n\to\infty} \frac{2\ln n}{n} = 2\lim_{n\to\infty} \frac{\ln n}{n} = 2\cdot 0 = 0$ (from formula 1).

(2) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} \left(n^{1/n}\right)^2 = \left(\lim_{n\to\infty} n^{1/n}\right)^2 = (1)^2 = 1$ (from formula 2).

(3) $\lim_{n\to\infty} \sqrt[n]{3n} = \lim_{n\to\infty} (3n)^{1/n} = \lim_{n\to\infty} \left(3^{1/n} \cdot n^{1/n}\right) = \lim_{n\to\infty} 3^{1/n} \cdot \lim_{n\to\infty} n^{1/n} = 1 \cdot 1 = 1$ (from formula 3 with $x = 3$ and formula 2).

(4) $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$ (from formula 4 with $x = -\frac{1}{2}$)

(5) $\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(\frac{n}{n} + \frac{-2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2}$ (from formula 5 with $x = -2$).

(6) $\lim_{n\to\infty} \frac{100^n}{n!} = 0$ (from formula 6 with $x = 100$).

Bounded Sequences

Definition: (bounded sequence)

A sequence $\{a_n\}$ is called bounded if there exists a real number $K > 0$ such that

$$|a_n| \le K \text{ for all } n \ge 1.$$

Definition:

(1) A sequence $\{a_n\}$ is called bounded from above if there exists a number M such that

$$a_n \le M \text{ for all } n \ge 1.$$

The number M is an upper bound for $\{a_n\}$.

(2) A sequence $\{a_n\}$ is called bounded from below if there exists a number m such that

$$a_n \ge m \text{ for all } n \ge 1.$$

The number m is an lower bound for $\{a_n\}$.

(3) A sequence $\{a_n\}$ is called bounded if bounded from above and below.

Example (6):

State whether the following sequence bounded from above, bounded from below, bounded or neither ?

(1) $1,2,3,\ldots,n,\ldots$ (2) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$

Solution:

(1) The sequence $1,2,3,\ldots,n,\ldots$ is bounded from below and lower bound is 1. This sequence is not bounded from above and so the sequence is not bounded.

(2) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$ is bounded from below and lower bound is $\frac{1}{2}$. Also the sequence is bounded from above because

$$n < n+1 \Rightarrow \frac{n}{n+1} < 1$$

and has upper bound 1. Since the sequence is bounded from below and bounded from above, the sequence $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$ is bounded.

Theorem (5):

If the sequence $\{a_n\}$ converges, then it is bounded.

Increasing and Decreasing Sequences

Definition:

(1) A sequence $\{a_n\}$ is called increasing sequence (nondecreasing sequence) if

$$a_n \le a_{n+1} \text{ for all } n \ge 1.$$

(2) A sequence $\{a_n\}$ is called decreasing sequence (nonincreasing sequence) if

$$a_n \ge a_{n+1} \text{ for all } n \ge 1.$$

(3) A sequence $\{a_n\}$ is called monotonic sequence if it is increasing or decreasing sequence.

Example (7):

State whether the following sequence increasing, decreasing or neither ?

(1) $1,2,3,\ldots,n,\ldots$ (2) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$ (3) $\left\{\frac{1}{(n+1)!}\right\}_{n=1}^{\infty}$

Solution:

(1) $1,2,3,\ldots,n,\ldots$, $a_n = n$

$\because n < n+1 \Rightarrow a_n < a_{n+1}$.

Then the sequence $1,2,3,\ldots,n,\ldots$ is increasing.

(2) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$, $a_n = \frac{n}{n+1}$

$$\because a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} \Rightarrow \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)} \Rightarrow \frac{1}{(n+1)(n+2)} > 0$$

. $\because a_{n+1} - a_n > 0 \Rightarrow \because a_{n+1} > a_n$.

Then the sequence $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{n}{n+1},\ldots$ is increasing.

(3) $\left\{\frac{1}{(n+1)!}\right\}_{n=1}^{\infty}$

$$\because \frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+2)!}}{\frac{1}{(n+1)!}} = \frac{(n+1)!}{(n+2)!} = \frac{(n+1)!}{(n+2)(n+1)!} = \frac{1}{n+2} < 1$$

$$\therefore \frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < a_n .$$

Then the sequence $\left\{\frac{1}{(n+1)!}\right\}_{n=1}^{\infty}$ is decreasing.

Theorem (6):

An increasing sequence of real numbers converges if and only if it is bounded from above.

Infinite Series

An infinite series is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Definition:

Given a sequence of numbers $\{a_n\}$, an expression of the form

$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

is an infinite series. The number a_n is the n th term of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \\
&\vdots
\end{aligned}$$

is the sequence of partial sums of the series, the number s_n being the n th partial sum.

Definition:

If the sequence of partial sums $\{s_n\}$ of the series converges to a limit L, that is,

$$\lim_{n \to \infty} s_n = L$$

we say that the series converges and that its sum is L. In this case, we also write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L .$$

If the sequence of partial sums $\{s_n\}$ of the series does not converge, we say that the series diverges.

Example:

Prove that the series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\,.$$

Converges and find its sum.

Solution:

$\because\ a_n=\dfrac{1}{n(n+1)}$

then the n th term a_n can be written as

$$a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

$$\begin{aligned}\therefore\ s_n &= a_1+a_2+a_3+\cdots+a_n\\ &=(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\cdots+(\frac{1}{\mathrm{n}}-\frac{1}{n+1})\\ &=1-\frac{1}{n+1}\end{aligned}$$

so $\lim\limits_{n\to\infty} s_n=\lim\limits_{n\to\infty}(1-\dfrac{1}{n+1})=1$. Then the series is convergent and its limit 1.

Example:

State whether the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}$ convergent or divergent ?

Solution:

$$\begin{aligned}
&\because s_1 = a_1 = 1,\\
&s_2 = a_1 + a_2 = 1-1 = 0,\\
&s_3 = a_1 + a_2 + a_3 = 1-1+1 = 1,\\
&s_4 = a_1 + a_2 + a_3 + a_4 = 1-1+1-1 = 0\\
&\quad\vdots\\
&s_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}$$

then, $\lim\limits_{n\to\infty} s_n$ does not exist and so the series is divergent.

Geometric Series

Definition:

Geometric series are series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum\limits_{n=0}^{\infty} ar^n$.

Theorem:

If $|r| < 1$, then geometric series $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum\limits_{n=1}^{\infty} ar^{n-1}$ converges to $\dfrac{a}{1-r}$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example:

State whether the following series convergent or divergent . If a series converges, find its sum ?

(1) $2+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\cdots+\frac{2}{3^{n-1}}+\cdots$

(2) $\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$

(3) $\sum_{n=0}^{\infty}\frac{(-1)^n 5}{4^n}$

(4) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$

Solution:

(1) $2+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\cdots+\frac{2}{3^{n-1}}+\cdots=2\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n-1}}+\cdots\right)=\sum_{n=1}^{\infty}2\frac{1}{3^{n-1}}.$

This series is a geometric series with $a=2$ and $r=\frac{1}{3}$.

$\because\ r=\frac{1}{3}<1 \Rightarrow$ the series is convergent and its sum $\frac{a}{1-r}=\frac{2}{1-1/3}=\frac{6}{2}=3$.

(2) $\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{9}\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{n-1}}+\cdots\right)=\sum_{n=1}^{\infty}\frac{1}{9}\frac{1}{3^{n-1}}$.

This series is a geometric series with $a=\frac{1}{9}$ and $r=\frac{1}{3}$.

$\because\ r=\frac{1}{3}<1 \Rightarrow$ the series is convergent and its sum $\frac{a}{1-r}=\frac{1/9}{1-1/3}=\frac{1}{6}$.

(3) $\sum_{n=0}^{\infty}\frac{(-1)^n 5}{4^n}=5-\frac{5}{4}+\frac{5}{4^2}-\frac{5}{4^3}+\cdots$

This series is a geometric series with $a=5$ and $r=-\frac{1}{4}$.

$\because\ |r|=|-\frac{1}{4}|=\frac{1}{4}<1 \Rightarrow$ the series is convergent and its sum $\frac{a}{1-r}=\frac{5}{1+1/4}=4$.

(4) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n=\frac{3}{2}+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\cdots$

This series is a geometric series with $a=\frac{3}{2}$ and $r=\frac{3}{2}$.

$\because r == \frac{3}{2} > 1 \Rightarrow$ the series is divergent.

Theorem:

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The n th-term test for divergence:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exists or is different from zero.

Example:

The following are all examples of divergent series:

(1) $\sum_{n=1}^{\infty} n^2$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$.

(2) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.

(3) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

(4) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1}$ does not exist.

Theorem:

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

(1) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$ (Sum Rule).

(2) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = A - B$ (Difference Rule).

(3) $\sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n = kA$ (any number k).

Remark:

1- Every nonzero constant multiple of a divergent series diverges.

2- If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n+b_n)$ and $\sum_{n=1}^{\infty}(a_n-b_n)$ both diverge.

Note that:

Remember that $\sum_{n=1}^{\infty}(a_n+b_n)$ can converge when $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$ both diverge. For example, $\sum_{n=1}^{\infty} a_n = 1+1+1+1+\cdots$ and $\sum_{n=1}^{\infty} b_n = (-1)+(-1)+(-1)+(-1)+\cdots$ diverge, whereas $\sum_{n=1}^{\infty}(a_n+b_n) = 0+0+0+\cdots$ converges to 0.

Example:

Find the sum of the series $\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}}$.

Solution:

$$\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{3^{n-1}}{6^{n-1}}-\frac{1}{6^{n-1}}\right) = \sum_{n=1}^{\infty}\left(\left(\frac{3}{6}\right)^{n-1}-\frac{1}{6^{n-1}}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}-\frac{1}{6^{n-1}}\right).$$

Then two series $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty}\frac{1}{6^{n-1}}$ converge because $r=\frac{1}{2}<1$ and $r=\frac{1}{6}<1$ respectively.

$$\therefore \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = \frac{1}{1-1/2} = 2 \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{1}{6^{n-1}} = \frac{1}{1-1/6} = \frac{6}{5}.$$

$$\therefore \sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}-\frac{1}{6^{n-1}}\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} - \sum_{n=1}^{\infty}\frac{1}{6^{n-1}} = 2-\frac{6}{5} = \frac{4}{5}.$$

Exercises

(1) Determine if the geometric series converges or diverges. If a series converges, find its sum.

(i) $3+\frac{3}{4}+\cdots+\frac{3}{4^{n-1}}+\cdots$

(ii) $1+\frac{e}{3}+\left(\frac{e}{3}\right)^2+\cdots+\left(\frac{e}{3}\right)^{n-1}+\cdots$

(iii) $0.37+0.0037+\cdots+\frac{37}{(100)^n}+\cdots$

(iv) $0.628+0.000628+\cdots+\frac{628}{(1000)^n}+\cdots$

(2) State whether the following series convergent or divergent . If a series converges, find its sum

(i) $\sum_{n=1}^{\infty}\left[\left(\frac{1}{4}\right)^n+\left(\frac{3}{4}\right)^n\right]$ (ii) $\sum_{n=1}^{\infty}\left[\left(\frac{3}{2}\right)^n+\left(\frac{2}{3}\right)^n\right]$

(iii) $\sum_{n=1}^{\infty}\left[\frac{1}{8^n}+\frac{1}{n(n+1)}\right]$ (iv) $\sum_{n=1}^{\infty}\left[\frac{1}{n(n+1)}-\frac{4}{n}\right]$.

(3) State, why the following series is divergent

(1) $\sum_{n=1}^{\infty}\frac{3n}{5n-1}$ (2) $\sum_{n=1}^{\infty}\frac{1}{1+(0.3)^n}$ (3) $\sum_{n=1}^{\infty}n\sin\left(\frac{1}{n}\right)$ (4) $\sum_{n=1}^{\infty}\ln\left(\frac{2n}{7n-5}\right)$

The integral test

Theorem: (the integral test)

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N is positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ is

(1) convergent if the integral $\int_N^{\infty} f(x)dx$ is convergent.

(2) divergent if the integral $\int_N^{\infty} f(x)dx$ is divergent.

Remark:

(1) The function f is increasing on interval I if $f'(x) > 0 \ \forall x \in I$.
(2) The function f is decreasing on interval I if $f'(x) < 0 \ \forall x \in I$.

Example:

State whether the following series convergent or divergent

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$ (2) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

Solution:

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$

Let $a_n = f(n) = \frac{1}{n}$. Then $f(x) = \frac{1}{x}$, for all $x \geq 1$, is positive and continuous.

$\because f'(x) = -\dfrac{1}{x^2} \Rightarrow f'(x) < 0 \;\; \forall\, x \geq 1$. Then the function f is decreasing.

Then we can use the integral test.

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}\left[\ln x\right]_1^t = \lim_{t\to\infty}\left[\ln t - \ln 1\right] = \lim_{t\to\infty}\left[\ln t\right] = \infty$$

then the integral diverges and so the series $\sum_{n=1}^{\infty}\dfrac{1}{n}$ is divergent.

(2) $\sum_{n=1}^{\infty}\dfrac{1}{1+n^2}$

Let $a_n = f(n) = \dfrac{1}{1+n^2}$. Then $f(x) = \dfrac{1}{1+x^2}$, for all $x \geq 1$, is positive and continuous.

$\because f'(x) = -\dfrac{2x}{\left(1+x^2\right)^2} \Rightarrow f'(x) < 0 \;\; \forall\, x \geq 1$. Then the function f is decreasing. Then we can use the integral test.

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty}\int_1^t \frac{1}{1+x^2}dx = \lim_{t\to\infty}\left[\tan^{-1}x\right]_1^t = \lim_{t\to\infty}\left[\tan^{-1}t - \tan^{-1}1\right] = \lim_{t\to\infty}\left[\tan^{-1}t - \frac{\pi}{4}\right]$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{4}$$

then the integral converges and so the series $\sum_{n=1}^{\infty}\dfrac{1}{1+n^2}$ is convergent.

<u>Definition</u> (P-series)

The series $\sum_{n=1}^{\infty}\dfrac{1}{n^p}$ is

(1) convergent if $p > 1$.

(2) divergent if $p \leq 1$.

Example:

(1) The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $p = 2 > 1$.

(2) The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges because $p = 3/2 > 1$.

(3) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because $p = 1/2 < 1$.

Exercises

Which of the series converge, and which diverge

(1) $\sum_{n=1}^{\infty} \frac{1}{10^n}$

(2) $\sum_{n=1}^{\infty} e^{-n}$

(3) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

(4) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

(5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)}$

(6) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

The comparison test

Theorem: (The comparison test)

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} c_n$ and $\sum_{n=1}^{\infty} d_n$ be series with nonnegative terms. Suppose that for some integral N

$$d_n \le a_n \le c_n \quad \text{for all} \quad n > N \ .$$

(i) If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) If $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Example:

Which of the series converge, and which diverge

(1) $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ (2) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (3) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Solution:

(1) $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$\because$ $5n > 5n-1 \Rightarrow \frac{1}{5n} < \frac{1}{5n-1} \Rightarrow \frac{5}{5n} < \frac{5}{5n-1} \Rightarrow \frac{1}{n} < \frac{5}{5n-1} \quad \forall \ n \ge 1$

$\because$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $p=1$ (p-series), then $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges.

(2) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\because\ \ln n < n \quad \forall\ n \ge 2 \Rightarrow \dfrac{1}{n} < \dfrac{1}{\ln n} \quad \forall\ n \ge 2$

$\because\ \sum_{n=2}^{\infty} \dfrac{1}{n}$ diverges because $p = 1$ (p-series), then $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges.

(3) $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^3}$

$\because\ \ln n < n \quad \forall\ n \ge 1 \Rightarrow \dfrac{\ln n}{n^3} < \dfrac{n}{n^3} \quad \Rightarrow \dfrac{\ln n}{n^3} < \dfrac{1}{n^2} \quad \forall\ n \ge 1$

$\because\ \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges because $p = 2 > 1$ (p-series), then $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^3}$ converges.

<u>Theorem:</u> (Limit comparison test)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer).

1. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
3. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Example:</u>

Which of the following series converge, and which diverge

(1) $\sum_{n=1}^{\infty} \dfrac{2n+1}{n^2+2n+1}$ (2) $\sum_{n=1}^{\infty} \dfrac{3n+1}{4n^3+n^2-2}$ (3) $\sum_{n=1}^{\infty} \dfrac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$

(4) $\sum_{n=1}^{\infty} \dfrac{1}{2^n-1}$ (5) $\sum_{n=2}^{\infty} \dfrac{1+n\ln n}{n^2+5}$ (6) $\sum_{n=1}^{\infty} \dfrac{3n^2+5n}{2^n(n^2+1)}$

Solution:

(1) $\sum_{n=1}^{\infty}\frac{2n+1}{n^2+2n+1}$

$a_n=\frac{2n+1}{n^2+2n+1}$. Consider $b_n=\frac{1}{n}$.

The series $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent series because $p=1$ (p-series).

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{2n+1}{n^2+2n+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{2n^2+n}{n^2+2n+1}=\lim_{n\to\infty}\frac{2+\frac{1}{n}}{1+\frac{2}{n}+\frac{1}{n^2}}=\frac{2}{1}=2>0$$

$\because$ $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty}\frac{2n+1}{n^2+2n+1}$ is divergent.

(2) $\sum_{n=1}^{\infty}\frac{3n+1}{4n^3+n^2-2}$

$a_n=\frac{3n+1}{4n^3+n^2-2}$. Consider $b_n=\frac{1}{n^2}$.

The series $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent series because $p=2>1$ (p-series).

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{3n+1}{4n^3+n^2-2}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{3n^3+n^2}{4n^3+n^2-2}=\lim_{n\to\infty}\frac{3+\frac{1}{n}}{4+\frac{1}{n}-\frac{2}{n^3}}=\frac{3}{4}>0$$

$\because$ $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, then $\sum_{n=1}^{\infty}\frac{3n+1}{4n^3+n^2-2}$ is convergent.

(3) $\sum_{n=1}^{\infty}\frac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$

$a_n = \dfrac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$. Consider $b_n = \dfrac{1}{n^{5/2}}$.

The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{n^{5/2}}$ is convergent series because $p = 5/2 > 1$ (p-series).

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{8n+\sqrt{n}}{5+n^2+n^{7/2}}}{\dfrac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{8n^{7/2}+n^3}{5+n^2+n^{7/2}} = \lim_{n\to\infty} \frac{8+\dfrac{1}{n^{1/2}}}{\dfrac{5}{n^{7/2}}+\dfrac{1}{n^{3/2}}+1} = \frac{8}{1} = 8 > 0$$

$\because$ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{n^{5/2}}$ is convergent, then $\sum_{n=1}^{\infty} \dfrac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$ is convergent.

(4) $\sum_{n=1}^{\infty} \dfrac{1}{2^n-1}$

$a_n = \dfrac{1}{2^n-1}$. Consider $b_n = \dfrac{1}{2^n}$.

The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{2^n}$ (geometric series) is convergent series because $r = \dfrac{1}{2} < 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{2^n-1}}{\dfrac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\dfrac{1}{2^n}} = \frac{1}{1-0} = 1 > 0$$

$\because$ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{2^n}$ is convergent, then $\sum_{n=1}^{\infty} \dfrac{1}{2^n-1}$ is convergent.

(5) $\sum_{n=2}^{\infty} \dfrac{1+n\ln n}{n^2+5}$

$a_n = \dfrac{1+n\ln n}{n^2+5}$. Consider $b_n = \dfrac{1}{n}$.

The series $\sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=2}^{\infty} \dfrac{1}{n}$ is divergent series because $p=1$ (p-series).

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1+n\ln n}{n^2+5}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n+n^2\ln n}{n^2+5} = \lim_{n\to\infty} \frac{\dfrac{1}{n}+\ln n}{1+\dfrac{5}{n^2}} = \infty$$

$\because \sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is divergent, then $\sum\limits_{n=2}^{\infty} \dfrac{1+n\ln n}{n^2+5}$ is divergent.

(6) $\sum\limits_{n=1}^{\infty} \dfrac{3\text{n}^2+5n}{2^n(n^2+1)}$

$a_n = \dfrac{3\text{n}^2+5n}{2^n(n^2+1)}$. Consider $b_n = \dfrac{1}{2^n}$.

The series $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}$ (geometric series) is convergent series because $r = \dfrac{1}{2} < 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{3\text{n}^2+5n}{2^n(n^2+1)}}{\dfrac{1}{2^n}} = \lim_{n\to\infty} \frac{3\text{n}^2+5n}{n^2+1} = \lim_{n\to\infty} \frac{3+\dfrac{5}{n}}{1+\dfrac{1}{n^2}} = \frac{3}{1} = 3 > 0$$

$\because \sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}$ is convergent, then $\sum\limits_{n=1}^{\infty} \dfrac{3\text{n}^2+5n}{2^n(n^2+1)}$ is convergent.

Exercises

Which of the following series converge, and which diverge

(1) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+4}$ (2) $\sum_{n=2}^{\infty}\frac{1}{\sqrt{4n^3-5n}}$ (3) $\sum_{n=1}^{\infty}\frac{2n+n^2}{n^3+1}$

(4) $\sum_{n=1}^{\infty}\frac{n^5+4n^3+1}{2n^8+n^4+2}$ (5) $\sum_{n=1}^{\infty}\frac{1}{1+3^n}$ (6) $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$

(7) $\sum_{n=1}^{\infty}\tan\frac{1}{n}$ (8) $\sum_{n=2}^{\infty}\frac{\ln n}{n^4}$

The Ratio and Root Tests

Theorem: (The ratio test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \; .$$

Then

(1) The series converges if $L < 1$.
(2) The series diverges if $L > 1$ or L is an infinite.
(3) The test fails if $L = 1$.

Example:

Which of the following series converge, and which diverge

(1) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ (2) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ (3) $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{2n!}$ (4) $\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$

Solution:

(1) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{(n+1)!}}{\dfrac{3^n}{n!}} = \lim_{n\to\infty} \frac{\dfrac{3 \cdot 3^n}{(n+1)\cdot n!}}{\dfrac{3^n}{n!}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

then the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges.

(2) $\sum_{n=1}^{\infty} \frac{n^{n}}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^{n}}{n!}} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)\cdot n!}}{\frac{n^{n}}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n}}{n^{n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n} + \frac{1}{n}\right)^{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} = e > 1$$

then the series $\sum_{n=1}^{\infty} \frac{n^{n}}{n!}$ diverges.

(3) $\sum_{n=1}^{\infty} \frac{3^{n}(n!)^{2}}{2n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3^{n+1}((n+1)!)^{2}}{(2n+2)!}}{\frac{3^{n}(n!)^{2}}{2n!}} = \lim_{n\to\infty} \frac{\frac{3\cdot 3^{n}(n+1)^{2}(n!)^{2}}{(2n+2)(2n+1)\cdot 2n!}}{\frac{3^{n}(n!)^{2}}{2n!}} = \lim_{n\to\infty} \frac{3(n+1)^{2}}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{3(n+1)^{2}}{2(n+1)(2n+1)} = \lim_{n\to\infty} 3\frac{(n+1)}{2(2n+1)} = \frac{3}{2}\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{3}{4} < 1$$

then the series $\sum_{n=1}^{\infty} \frac{3^{n}(n!)^{2}}{2n!}$ converges.

(4) $\sum_{n=1}^{\infty} \frac{2^{n}+5}{3^{n}}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}=\lim_{n\to\infty}\frac{1}{3}\frac{2^{n+1}+5}{2^n+5}=\frac{1}{3}\lim_{n\to\infty}\frac{\frac{2^{n+1}}{2^n}+\frac{5}{2^n}}{1+\frac{5}{2^n}}=\frac{1}{3}\lim_{n\to\infty}\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}=\frac{1}{3}\frac{2}{1}=\frac{2}{3}<1$$

then the series $\sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$ converges.

Theorem: (The root test)

Let $\sum_{n=1}^{\infty}a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty}\sqrt[n]{a_n}=L\ .$$

Then

(1) The series converges if $L<1$.
(2) The series diverges if $L>1$ or L is an infinite.
(3) The test fails if $L=1$.

Example:

Which of the following series converge, and which diverge

(1) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ (2) $\sum_{n=1}^{\infty}\frac{2^n}{n^3}$ (3) $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^n$

Solution:

(1) $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{2^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n^2}}{2}=\lim_{n\to\infty}\frac{(\sqrt[n]{n})^2}{2}=\frac{1}{2}(\lim_{n\to\infty}\sqrt[n]{n})^2=\frac{1}{2}\cdot(1)^2=\frac{1}{2}<1$$

then the series $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ converges.

(2) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^3}} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^3}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^3} = 2\frac{1}{(\lim_{n\to\infty} \sqrt[n]{n})^3} = 2 \cdot \frac{1}{1^3} = 2 > 1$$

then the series $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges.

(3) $\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n+1}\right)^n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

then the series $\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n$ converges.

Exercises

Which of the following series converge, and which diverge

(1) $\sum_{n=1}^{\infty} \frac{3n+1}{2^n}$ (2) $\sum_{n=1}^{\infty} \frac{5^n}{n(3^{n+1})}$ (3) $\sum_{n=1}^{\infty} \frac{(100)^n}{n!}$

(4) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ (5) $\sum_{n=1}^{\infty} \frac{n!}{(n+1)^5}$ (6) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

(7) $\sum_{n=1}^{\infty} \frac{1}{n^n}$ (8) $\sum_{n=2}^{\infty} \frac{5^{n+1}}{(\ln n)^n}$ (9) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Alternating Series, Absolute and Conditional Convergence

Theorem: The alternating series test (Leibniz's test)

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied

1- The u_n 's are all positive.

2- The positive u_n 's are decreasing:

$$u_n \geq u_{n+1} \text{ for all } n.$$

3- $\lim_{n\to\infty} u_n = 0$.

Example:

Which of the following series converge, and which diverge

(1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ (2) $\sum_{n=4}^{\infty}(-1)^{n+1}\frac{10n}{n^2+16}$

Solution:

(1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

$u_n = \frac{1}{n}$, $n \geq 1$

(i) u_n 's are all positive

(ii) if $f(n) = \frac{1}{n} \Rightarrow f(x) = \frac{1}{x}$, $x \geq 1 \Rightarrow f'(x) = -\frac{1}{x^2} \leq 0 \ \forall \ x \geq 1$

then $f(x)$ is decreasing and so u_n is decreasing.

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

then all three conditions are satisfied and so $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges.

(2) $\sum_{n=4}^{\infty} (-1)^{n+1} \frac{10n}{n^2+16}$

$u_n = \frac{10n}{n^2+16}$, $n \geq 4$

(i) u_n 's are all positive

(ii) if

$f(n) = \frac{10n}{n^2+16} \Rightarrow f(x) = \frac{10x}{x^2+16},\ x \geq 4 \Rightarrow f'(x) = \frac{10(x^2+16) - 2x(10x)}{(x^2+16)^2}$

$f'(x) = \frac{10(16-x^2)}{(x^2+16)^2} \leq 0 \ \ \forall\ x \geq 4.$

then $f(x)$ is decreasing and so u_n is decreasing.

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{10n}{n^2+16} = \lim_{n\to\infty} \frac{\frac{10}{n}}{1+\frac{16}{n^2}} = \frac{0}{1} = 0$

then all three conditions are satisfied and so $\sum_{n=4}^{\infty} (-1)^{n+1} \frac{10n}{n^2+16}$ converges.

<u>**Definition:**</u>

A series $\sum_{n=1}^{\infty} a_n$ converges absolutely (is absolutely convergent) is if the series $\sum_{n=1}^{\infty} |a_n|$ converges .

Theorem: (The absolute convergence test)

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Remark:

The converse statement of the above theorem is false. For example;

in above example we show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is converges, but the series $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (p-series and $p=1$).

Example:

Prove that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is absolutely convergent.

Solution:

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\because$ $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent because $p=2>1$ (p-series), then the series $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right|$ is convergent and so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is absolutely convergent.

Definition:

The series $\sum_{n=1}^{\infty}a_n$ is conditional convergence if the series $\sum_{n=1}^{\infty}a_n$ converges but the series $\sum_{n=1}^{\infty}|a_n|$ diverges.

Example:

Prove that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is conditional convergence.

Solution:

$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent because $p = \frac{1}{2} \leq 1$ (p-series).

Then the series is not absolutely convergent. Now we discuss the convergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

$u_n = \frac{1}{\sqrt{n}}$, $n \geq 1$

(i) u_n 's are all positive

(ii) if $f(n) = \frac{1}{\sqrt{n}} \Rightarrow f(x) = \frac{1}{\sqrt{x}}$, $x \geq 1 \Rightarrow f'(x) = -\frac{1}{2}\frac{1}{x^{3/2}} \leq 0 \ \forall \ x \geq 1$

then $f(x)$ is decreasing and so u_n is decreasing.

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

then all three conditions are satisfied and so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ converges. Then the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is conditional convergence.

<u>Exercises</u>

State whether the following series absolutely convergence, conditional convergence or divergent ?

(1) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2+7}$

(2) $\sum_{n=1}^{\infty}(-1)^{n}\left(1+e^{-n}\right)$

(3) $\sum_{n=1}^{\infty}(-1)^{n}\frac{e^{2n}+1}{e^{2n}-1}$

(4) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{2n+1}}$

(5) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\ln(n+1)}$

(6) $\sum_{n=2}^{\infty}(-1)^{n}\frac{n}{\ln n}$

(7) $\sum_{n=1}^{\infty}(-1)^{n}\frac{5}{n^3+1}$

(8) $\sum_{n=1}^{\infty}\frac{(-10)^n}{n!}$

(9) $\sum_{n=1}^{\infty}(-1)^{n}n\sin\frac{1}{n}$

(10) $\sum_{n=1}^{\infty}(-1)^{n}\frac{1+4^n}{1+3^n}$

Power Series

Definition:

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \tag{1}$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Remark:

Recall that the Ratio Test applies to series with nonnegative terms.

Example:

For what values of x do the following power series converge ?

(1) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

(2) $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = 1 - \frac{1}{2}(x-2) + \left(\frac{1}{2}\right)^2 (x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$

(3) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$

(4) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots$

(5) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

(6) $\sum_{n=0}^{\infty} n! x^n = 1 + 1! x + 2! x^2 + \cdots + n! x^n + \cdots$

Solution:

(1) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

This is geometric series with first term 1 and ratio $r = x$, then the series converges for $|x| < 1 \Rightarrow -1 < x < 1$ and its sum $\frac{a}{1-r} = \frac{1}{1-x}$.

(2) $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = 1 - \frac{1}{2}(x-2) + \left(\frac{1}{2}\right)^2 (x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$

This is geometric series with first term 1 and ratio $r = -\frac{1}{2}(x-2)$, then the series converges for $\left|-\frac{1}{2}(x-2)\right| < 1 \Rightarrow |x-2| < 2 \Rightarrow -2 < x-2 < 2 \Rightarrow 0 < x < 4$ and its sum $\frac{a}{1-r} = \frac{1}{1+\frac{1}{2}(x-2)} = \frac{2}{2+(x-2)} = \frac{2}{x}$.

(3) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$

Apply the ratio test to the series $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{x^n}{n}\right| = \sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty} \left|\frac{x\,n}{n+1}\right| = \lim_{n\to\infty} \frac{|x|\,n}{n+1} = |x|$$

The series $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{x^n}{n}\right| = \sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ is converges for $|x| < 1$. Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ is absolutely convergence for $|x| < 1$.

The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ diverges for $|x|>1$, because the condition

$$\lim_{n\to\infty}\frac{x^n}{n}=\frac{\infty}{\infty}\ \ (\lim_{n\to\infty}x^n=\infty\ \text{ if } |x|>1)$$

$$\therefore\ \lim_{n\to\infty}\frac{x^n}{n}=\lim_{n\to\infty}\frac{x^n\ln x}{1}=\infty\neq 0.$$

If $x=1$, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges because it satisfies all three conditions of the alternating series test.

If $x=-1$, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n}=\sum_{n=1}^{\infty}(-1)^{2n-1}\frac{1}{n}=\sum_{n=1}^{\infty}(-1)^{2n}(-1)^{-1}\frac{1}{n}=-\sum_{n=1}^{\infty}\frac{1}{n}$

diverges because, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (*p*-series).

Then from above the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for $-1<x\leq 1$.

(4) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+\cdots$

Apply the ratio test to the series $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right|=\sum_{n=1}^{\infty}\left|\frac{x^{2n-1}}{2n-1}\right|$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{2n+1}}{2n+1}}{\frac{x^{2n-1}}{2n-1}}\right|=\lim_{n\to\infty}\left|\frac{x^2(2n-1)}{2n+1}\right|=\lim_{n\to\infty}\frac{x^2(2n-1)}{2n+1}=x^2$$

The series $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right|=\sum_{n=1}^{\infty}\left|\frac{x^{2n-1}}{2n-1}\right|$ is converges for $x^2<1\Rightarrow|x|<1$.

Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ is absolutely convergence for $|x|<1$.

The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ diverges for $x^2>1 \Rightarrow |x|>1$, because the condition

$$\lim_{n\to\infty}\frac{x^{2n-1}}{2n-1}=\frac{\infty}{\infty} \quad (\lim_{n\to\infty}x^{2n-1}=\infty \text{ if } |x|>1)$$

$$\therefore \lim_{n\to\infty}\frac{x^{2n-1}}{2n-1}=\lim_{n\to\infty}\frac{x^{2n-1}(2)\ln x}{2}=\lim_{n\to\infty}x^{2n-1}\ln x=\infty\neq 0.$$

If $x=1$, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}$ converges because it satisfies all three conditions of the alternating series test.

If $x=-1$, the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^{2n-1}}{2n-1}=\sum_{n=1}^{\infty}(-1)^{3n-2}\frac{1}{2n-1}=\sum_{n=1}^{\infty}(-1)^{n}(-1)^{2n-2}\frac{1}{2n-1}=\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{2n-1}$$

converges because it satisfies all three conditions of the alternating series test.

Then from above the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ converges for $-1\leq x\leq 1$.

(5) $\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$

Apply the ratio test to the series $\sum_{n=0}^{\infty}|u_n|=\sum_{n=0}^{\infty}\left|\frac{x^n}{n!}\right|$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=\lim_{n\to\infty}\frac{|x|}{n+1}=0 \text{ for every } x.$$

Then the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is absolutely convergence for all x.

(6) $\sum_{n=0}^{\infty}n!x^n=1+1!x+2!x^2+\cdots+n!x^n+\cdots$

Apply the ratio test to the series $\sum_{n=0}^{\infty}|u_n| = \sum_{n=0}^{\infty}|n!x^n|$

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}|(n+1)x| = \lim_{n\to\infty}(n+1)|x| = \infty$ for every x except $x = 0$.

Then the series $\sum_{n=0}^{\infty} n!\,x^n$ diverges for all x except $x = 0$.

The Radius of Convergence of a Power Series

Theorem:

The convergence of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is described by one of the following three cases:

1- There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2- The series converges absolutely for every x ($R = \infty$).

3- The series converges at $x = a$ and diverges elsewhere ($R = 0$).

R is called the radius of convergence of the power series, and the interval of radius R centered at $x = a$ is called the interval of convergence.

Remark:

The interval of convergence may be open, closed, or half-open, depending on the particular series. At points x with $|x-a| < R$, the series converges absolutely. If the series converges for all values of x, we say its radius of convergence is infinite. If it converges only at $x = a$, we say its radius of convergence is zero.

Example:

Find the series' radius and interval of convergence of the following power series.

(1) $\sum_{n=0}^{\infty}(x+5)^n$ (2) $\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n}$ (3) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$ (4) $\sum_{n=1}^{\infty}\frac{x^n}{n\sqrt{n}\,3^n}$

(5) $\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}$ (6) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ (7) $\sum_{n=1}^{\infty} n^n x^n$.

Solution:

(1) $\sum_{n=0}^{\infty}(x+5)^n = 1+(x+5)+(x+5)^2+(x+5)^2+\cdots$

This is geometric series with first term 1 and ratio $r = x+5$, then the series converges for $|x+5|<1$ $(R=1) \Rightarrow -1<x+5<1 \Rightarrow -6<x<-4$.

Then the radius $R=1$and the interval of convergence is $-6<x<-4$.

(2) $\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n}$

Apply the ratio test to the series $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\left|\frac{(3x-2)^n}{n}\right|$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)n}{n+1}\right| = |3x-2|\lim_{n\to\infty}\frac{n}{n+1} = |3x-2|$$

The series $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\left|\frac{(3x-2)^n}{n}\right|$ is converges for $|3x-2|<1$. Then the series $\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n}$ is absolutely convergence for $|3x-2|<1$.

$$|3x-2|<1 \Rightarrow |x-\frac{2}{3}|<\frac{1}{3} \quad (R=\frac{1}{3}) \Rightarrow -\frac{1}{3}<x-\frac{2}{3}<\frac{1}{3} \Rightarrow \frac{1}{3}<x<1 .$$

When $x=\frac{1}{3}$, the series $\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(3\frac{1}{3}-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(1-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

converges .

When $x=1$, the series $\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(3-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(1)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$

diverges .

Then the radius is $R = \dfrac{1}{3}$ and the interval of convergence is $\dfrac{1}{3} \le x < 1$.

(3) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n} = 1 + \frac{(x-2)}{10} + \frac{(x-2)^2}{10^2} + \frac{(x-2)^3}{10^3} + \cdots$

This is geometric series with first term 1 and ratio $r = \dfrac{(x-2)}{10}$, then the series converges for $|\dfrac{(x-2)}{10}| < 1 \Rightarrow |x-2| < 10 \ \ (R = 10) \ \Rightarrow -10 < x - 2 < 10 \Rightarrow -8 < x < 12$.

Then the radius $R = 10$ and the interval of convergence is $-8 < x < 12$.

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\ 3^n}$

Apply the ratio test to the series $\displaystyle\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left| \frac{x^n}{n\sqrt{n}\ 3^n} \right|$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{x^{n+1}}{(n+1)\sqrt{n+1}\ 3^{n+1}}}{\dfrac{x^n}{n\sqrt{n}\ 3^n}} \right| = \lim_{n\to\infty} \left| \frac{xn\sqrt{n}}{3(n+1)\sqrt{n+1}} \right| = \frac{|x|}{3} \lim_{n\to\infty} \frac{n\sqrt{n}}{(n+1)\sqrt{n+1}}$$

$$= \frac{|x|}{3} \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x|}{3} \lim_{n\to\infty} \frac{n}{n+1} \cdot \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \frac{|x|}{3}$$

The series $\displaystyle\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left| \frac{x^n}{n\sqrt{n}\ 3^n} \right|$ is converges for $\dfrac{|x|}{3} < 1$. Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\ 3^n}$ is absolutely convergence for $\dfrac{|x|}{3} < 1 \Rightarrow |x| < 3 \ \ (R = 3) \ \Rightarrow \ -3 < x < 3$.

When $x = -3$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\ 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}\ 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (3)^n}{n\sqrt{n}\ 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$ is converges because it satisfies all three conditions of the alternating series test.

When $x=3$, the series $\sum_{n=1}^{\infty}\frac{x^n}{n\sqrt{n}\ 3^n}=\sum_{n=1}^{\infty}\frac{3^n}{n\sqrt{n}\ 3^n}=\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is converges because $p=\frac{3}{2}>1$ (p-series). Then radius is $R=3$ and the interval of convergence is $-3\le x\le 3$.

(5) $\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}$

Apply the ratio test to the series $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\left|\frac{(x-1)^n}{\sqrt{n}}\right|$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{\sqrt{n+1}}}{\frac{(x-1)^n}{\sqrt{n}}}\right|=\lim_{n\to\infty}\left|\frac{(x-1)\sqrt{n}}{\sqrt{n+1}}\right|=|x-1|\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$=|x-1|\sqrt{\lim_{n\to\infty}\frac{n}{n+1}}=|x-1|$$

The series $\sum_{n=1}^{\infty}|u_n|=\sum_{n=1}^{\infty}\left|\frac{(x-1)^n}{\sqrt{n}}\right|$ is converges for $|x-1|<1$. Then the series $\sum_{n=1}^{\infty}\frac{x^n}{n\sqrt{n}\ 3^n}$ is absolutely convergence for $|x-1|<1$ $(R=1)$ $\Rightarrow$ $-1<x-1<1$ $\Rightarrow$ $0<x<2$.

When $x=0$, the series $\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is converges because it satisfies all three conditions of the alternating series test.

When $x=2$, the series $\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(2-1)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(1)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is diverges because $p=\frac{1}{2}<1$ (p-series). Then radius is $R=1$ and the interval of convergence is $0\le x<2$.

(6) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$

Apply the ratio test to the series $\sum_{n=0}^{\infty}|u_n| = \sum_{n=0}^{\infty}\left|\frac{(-1)^n x^n}{n!}\right| = \sum_{n=0}^{\infty}\left|\frac{x^n}{n!}\right|$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty}\frac{1}{n+1} = 0 \text{ for every } x \text{ .}$$

Then the series $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ is absolutely convergence for all x . Then the radius is $R=\infty$ and the series converges for all x .

(7) $\sum_{n=1}^{\infty} n^n x^n$

Apply the ratio test to the series $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}|n^n x^n|$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n+1)^n x}{n^n}\right| = |x| \lim_{n\to\infty}(n+1)\cdot\lim_{n\to\infty}\frac{(n+1)^n}{n^n}$$

$$= |x| \lim_{n\to\infty}(n+1)\cdot\lim_{n\to\infty}\frac{(n+1)^n}{n^n} = |x| \lim_{n\to\infty}(n+1)\cdot\lim_{n\to\infty}(\frac{n+1}{n})^n$$

$$= |x| \lim_{n\to\infty}(n+1)\cdot\lim_{n\to\infty}(1+\frac{1}{n})^n = |x| \cdot\infty\cdot e = \infty \text{ for every } x \text{ except } x=0.$$

Then the series $\sum_{n=1}^{\infty} n^n x^n$ diverges for all x except $x=0$. Then the radius is $R=0$ and the series converges only for $x=0$.

Chapter 1

LINEAR EQUATIONS

1.1 Introduction to linear equations

A *linear equation* in n unknowns $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are given real numbers.

For example, with x and y instead of x_1 and x_2, the linear equation $2x + 3y = 6$ describes the line passing through the points (3, 0) and (0, 2).

Similarly, with x, y and z instead of x_1, x_2 and x_3, the linear equation $2x + 3y + 4z = 12$ describes the plane passing through the points (6, 0, 0), (0, 4, 0), (0, 0, 3).

A *system* of m linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a family of linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

We wish to determine if such a system has a solution, that is to find out if there exist numbers $x_1, x_2, \cdots, x_n$ which satisfy each of the equations simultaneously. We say that the system is *consistent* if it has a solution. Otherwise the system is called *inconsistent*.

Note that the above system can be written concisely as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1,\, 2, \cdots, m.$$

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the *coefficient matrix* of the system, while the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is called the *augmented matrix* of the system.

Geometrically, solving a system of linear equations in two (or three) unknowns is equivalent to determining whether or not a family of lines (or planes) has a common point of intersection.

EXAMPLE 1.1.1 Solve the equation

$$2x + 3y = 6.$$

Solution. The equation $2x + 3y = 6$ is equivalent to $2x = 6 - 3y$ or $x = 3 - \frac{3}{2}y$, where y is arbitrary. So there are infinitely many solutions.

EXAMPLE 1.1.2 Solve the system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 0. \end{aligned}$$

Solution. We subtract the second equation from the first, to get $2y = 1$ and $y = \frac{1}{2}$. Then $x = y - z = \frac{1}{2} - z$, where z is arbitrary. Again there are infinitely many solutions.

EXAMPLE 1.1.3 Find a polynomial of the form $y = a_0 + a_1x + a_2x^2 + a_3x^3$ which passes through the points $(-3, -2)$, $(-1, 2)$, $(1, 5)$, $(2, 1)$.

Solution. When x has the values -3, -1, 1, 2, then y takes corresponding values -2, 2, 5, 1 and we get four equations in the unknowns a_0, a_1, a_2, a_3:

$$\begin{aligned}
a_0 - 3a_1 + 9a_2 - 27a_3 &= -2 \\
a_0 - a_1 + a_2 - a_3 &= 2 \\
a_0 + a_1 + a_2 + a_3 &= 5 \\
a_0 + 2a_1 + 4a_2 + 8a_3 &= 1.
\end{aligned}$$

This system has the unique solution $a_0 = 93/20$, $a_1 = 221/120$, $a_2 = -23/20$,
$a_3 = -41/120$. So the required polynomial is

$$y = \frac{93}{20} + \frac{221}{120}x - \frac{23}{20}x^2 - \frac{41}{120}x^3.$$

In [26, pages 33–35] there are examples of systems of linear equations which arise from simple electrical networks using Kirchhoff's laws for electrical circuits.

Solving a system consisting of a single linear equation is easy. However if we are dealing with two or more equations, it is desirable to have a systematic method of determining if the system is consistent and to find all solutions.

Instead of restricting ourselves to linear equations with rational or real coefficients, our theory goes over to the more general case where the coefficients belong to an arbitrary *field*. A *field* F is a set F which possesses operations of *addition* and *multiplication* which satisfy the familiar rules of rational arithmetic. There are ten basic properties that a field must have:

THE FIELD AXIOMS.

1. $(a + b) + c = a + (b + c)$ for all a, b, c in F;
2. $(ab)c = a(bc)$ for all a, b, c in F;
3. $a + b = b + a$ for all a, b in F;
4. $ab = ba$ for all a, b in F;
5. there exists an element 0 in F such that $0 + a = a$ for all a in F;
6. there exists an element 1 in F such that $1a = a$ for all a in F;

7. to every a in F, there corresponds an *additive inverse* $-a$ in F, satisfying
$$a + (-a) = 0;$$

8. to every non–zero a in F, there corresponds a *multiplicative inverse* a^{-1} in F, satisfying
$$aa^{-1} = 1;$$

9. $a(b + c) = ab + ac$ for all a, b, c in F;

10. $0 \neq 1$.

With standard definitions such as $a - b = a + (-b)$ and $\dfrac{a}{b} = ab^{-1}$ for $b \neq 0$, we have the following familiar rules:

$$
\begin{aligned}
-(a+b) &= (-a) + (-b), \quad (ab)^{-1} = a^{-1}b^{-1};\\
-(-a) &= a, \quad (a^{-1})^{-1} = a;\\
-(a-b) &= b - a, \quad (\frac{a}{b})^{-1} = \frac{b}{a};\\
\frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd};\\
\frac{a}{b}\frac{c}{d} &= \frac{ac}{bd};\\
\frac{ab}{ac} &= \frac{b}{c}, \quad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b};\\
-(ab) &= (-a)b = a(-b);\\
-\left(\frac{a}{b}\right) &= \frac{-a}{b} = \frac{a}{-b};\\
0a &= 0;\\
(-a)^{-1} &= -(a^{-1}).
\end{aligned}
$$

Fields which have only finitely many elements are of great interest in many parts of mathematics and its applications, for example to coding theory. It is easy to construct fields containing exactly p elements, where p is a prime number. First we must explain the idea of *modular addition* and *modular multiplication*. If a is an integer, we define $a \pmod{p}$ to be the *least remainder on dividing a by p*: That is, if $a = bp + r$, where b and r are integers and $0 \leq r < p$, then $a \pmod{p} = r$.

For example, $-1 \pmod{2} = 1,\ 3 \pmod{3} = 0,\ 5 \pmod{3} = 2$.

Then addition and multiplication mod p are defined by

$$\begin{aligned} a \oplus b &= (a+b) \,(\operatorname{mod} p) \\ a \otimes b &= (ab) \,(\operatorname{mod} p). \end{aligned}$$

For example, with $p = 7$, we have $3 \oplus 4 = 7 \,(\operatorname{mod} 7) = 0$ and $3 \otimes 5 = 15 \,(\operatorname{mod} 7) = 1$. Here are the complete addition and multiplication tables mod 7:

$\oplus$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

$\otimes$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

If we now let $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$, then it can be proved that $\mathbb{Z}_p$ forms a field under the operations of modular addition and multiplication mod p. For example, the additive inverse of 3 in $\mathbb{Z}_7$ is 4, so we write $-3 = 4$ when calculating in $\mathbb{Z}_7$. Also the multiplicative inverse of 3 in $\mathbb{Z}_7$ is 5 , so we write $3^{-1} = 5$ when calculating in $\mathbb{Z}_7$.

In practice, we write $a \oplus b$ and $a \otimes b$ as $a+b$ and ab or $a \times b$ when dealing with linear equations over $\mathbb{Z}_p$.

The simplest field is $\mathbb{Z}_2$, which consists of two elements 0, 1 with addition satisfying $1+1=0$. So in $\mathbb{Z}_2$, $-1 = 1$ and the arithmetic involved in solving equations over $\mathbb{Z}_2$ is very simple.

EXAMPLE 1.1.4 Solve the following system over $\mathbb{Z}_2$:

$$\begin{aligned} x + y + z &= 0 \\ x + z &= 1. \end{aligned}$$

Solution. We add the first equation to the second to get $y = 1$. Then $x = 1 - z = 1 + z$, with z arbitrary. Hence the solutions are $(x, y, z) = (1, 1, 0)$ and $(0, 1, 1)$.

We use $\mathbb{Q}$ and $\mathbb{R}$ to denote the fields of rational and real numbers, respectively. Unless otherwise stated, the field used will be $\mathbb{Q}$.

1.2 Solving linear equations

We show how to solve any system of linear equations over an arbitrary field, using the *GAUSS–JORDAN* algorithm. We first need to define some terms.

DEFINITION 1.2.1 (Row–echelon form) A matrix is in *row–echelon form* if

(i) all zero rows (if any) are at the bottom of the matrix and

(ii) if two successive rows are non–zero, the second row starts with more zeros than the first (moving from left to right).

For example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row–echelon form, whereas the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in row–echelon form.

The *zero* matrix of any size is always in row–echelon form.

DEFINITION 1.2.2 (Reduced row–echelon form) A matrix is in *reduced row–echelon form* if

1. it is in row–echelon form,
2. the leading (leftmost non–zero) entry in each non–zero row is 1,
3. all other elements of the column in which the leading entry 1 occurs are zeros.

For example the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are in reduced row–echelon form, whereas the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are not in reduced row–echelon form, but are in row–echelon form.

The *zero* matrix of any size is always in reduced row–echelon form.

Notation. If a matrix is in reduced row–echelon form, it is useful to denote the column numbers in which the leading entries 1 occur, by $c_1, c_2, \ldots, c_r$, with the remaining column numbers being denoted by $c_{r+1}, \ldots, c_n$, where r is the number of non–zero rows. For example, in the 4×6 matrix above, we have $r = 3$, $c_1 = 2$, $c_2 = 4$, $c_3 = 5$, $c_4 = 1$, $c_5 = 3$, $c_6 = 6$.

The following operations are the ones used on systems of linear equations and do not change the solutions.

DEFINITION 1.2.3 (Elementary row operations) There are three types of *elementary row operations* that can be performed on matrices:

1. Interchanging two rows:

   $R_i \leftrightarrow R_j$ interchanges rows i and j.

2. Multiplying a row by a non–zero scalar:

   $R_i \to tR_i$ multiplies row i by the non–zero scalar t.

3. Adding a multiple of one row to another row:

   $R_j \to R_j + tR_i$ adds t times row i to row j.

DEFINITION 1.2.4 *[Row equivalence]*Matrix A is *row–equivalent* to matrix B if B is obtained from A by a sequence of elementary row operations.

EXAMPLE 1.2.1 Working from left to right,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad R_2 \to R_2 + 2R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} \quad R_1 \to 2R_1 \quad \begin{bmatrix} 2 & 4 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} = B.$$

Thus A is row–equivalent to B. Clearly B is also row–equivalent to A, by performing the inverse row–operations $R_1 \to \frac{1}{2}R_1$, $R_2 \leftrightarrow R_3$, $R_2 \to R_2 - 2R_3$ on B.

It is not difficult to prove that if A and B are row–equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets – a solution of the one system is a solution of the other. For example the systems whose augmented matrices are A and B in the above example are respectively

$$\left\{ \begin{array}{rcl} x + 2y &=& 0 \\ 2x + y &=& 1 \\ x - y &=& 2 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} 2x + 4y &=& 0 \\ x - y &=& 2 \\ 4x - y &=& 5 \end{array} \right.$$

and these systems have precisely the same solutions.

1.3 The Gauss–Jordan algorithm

We now describe the *GAUSS–JORDAN ALGORITHM*. This is a process which starts with a given matrix A and produces a matrix B in reduced row–echelon form, which is row–equivalent to A. If A is the augmented matrix of a system of linear equations, then B will be a much simpler matrix than A from which the consistency or inconsistency of the corresponding system is immediately apparent and in fact the complete solution of the system can be read off.

<u>STEP 1</u>.

Find the first non–zero column moving from left to right, (column c_1) and select a non–zero entry from this column. By interchanging rows, if necessary, ensure that the first entry in this column is non–zero. Multiply row 1 by the multiplicative inverse of a_{1c_1} thereby converting a_{1c_1} to 1. For each non–zero element a_{ic_1}, $i > 1$, (if any) in column c_1, add $-a_{ic_1}$ times row 1 to row i, thereby ensuring that all elements in column c_1, apart from the first, are zero.

<u>STEP 2</u>. If the matrix obtained at Step 1 has its $2\text{nd}, \ldots, m\text{th}$ rows all zero, the matrix is in reduced row–echelon form. Otherwise suppose that the first column which has a non–zero element in the rows below the first is column c_2. Then $c_1 < c_2$. By interchanging rows below the first, if necessary, ensure that a_{2c_2} is non–zero. Then convert a_{2c_2} to 1 and by adding suitable multiples of row 2 to the remaing rows, where necessary, ensure that all remaining elements in column c_2 are zero.

The process is repeated and will eventually stop after r steps, either because we run out of rows, or because we run out of non–zero columns. In general, the final matrix will be in reduced row–echelon form and will have r non–zero rows, with leading entries 1 in columns $c_1, \ldots, c_r$, respectively.

EXAMPLE 1.3.1

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 2 & 2 & -2 & 5 \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_1 \to \tfrac{1}{2}R_1 \quad \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix} \quad R_3 \to R_3 - 5R_1 \quad \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{bmatrix}$$

$$R_2 \to \tfrac{1}{4}R_2 \quad \begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -\frac{15}{2} \end{bmatrix} \quad \left\{ \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 4R_2 \end{array} \right. \quad \begin{bmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{15}{2} \end{bmatrix}$$

$$R_3 \to \tfrac{-2}{15}R_3 \quad \begin{bmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 - \tfrac{5}{2}R_3 \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last matrix is in reduced row–echelon form.

REMARK 1.3.1 It is possible to show that a given matrix over an arbitrary field is row–equivalent to *precisely one* matrix which is in reduced row–echelon form.

A flow–chart for the Gauss–Jordan algorithm, based on [1, page 83] is presented in figure 1.1 below.

1.4 Systematic solution of linear systems.

Suppose a system of m linear equations in n unknowns $x_1, \cdots, x_n$ has augmented matrix A and that A is row–equivalent to a matrix B which is in reduced row–echelon form, via the Gauss–Jordan algorithm. Then A and B are $m \times (n+1)$. Suppose that B has r non–zero rows and that the leading entry 1 in row i occurs in column number c_i, for $1 \leq i \leq r$. Then

$$1 \leq c_1 < c_2 < \cdots, < c_r \leq n+1.$$

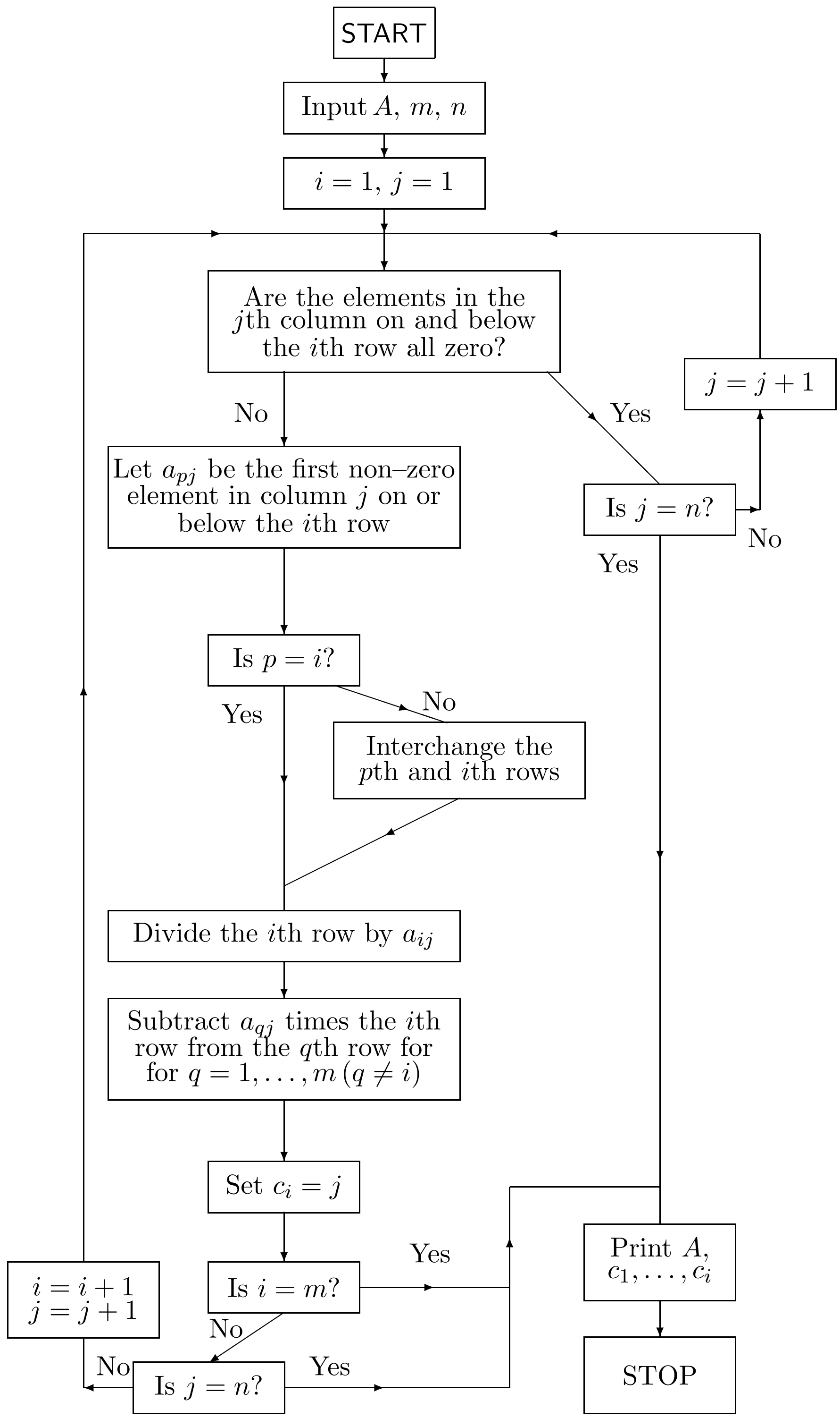


Figure 1.1: Gauss–Jordan algorithm.

Also assume that the remaining column numbers are $c_{r+1}, \cdots, c_{n+1}$, where

$$1 \leq c_{r+1} < c_{r+2} < \cdots < c_n \leq n+1.$$

Case 1: $c_r = n+1$. The system is inconsistent. For the last non–zero row of B is $[0, 0, \cdots, 1]$ and the corresponding equation is

$$0x_1 + 0x_2 + \cdots + 0x_n = 1,$$

which has no solutions. Consequently the original system has no solutions.

Case 2: $c_r \leq n$. The system of equations corresponding to the non–zero rows of B is consistent. First notice that $r \leq n$ here.

If $r = n$, then $c_1 = 1$, $c_2 = 2$, $\cdots$, $c_n = n$ and

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & d_1 \\ 0 & 1 & \cdots & 0 & d_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & d_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

There is a unique solution $x_1 = d_1$, $x_2 = d_2$, $\cdots$, $x_n = d_n$.

If $r < n$, there will be more than one solution (infinitely many if the field is infinite). For all solutions are obtained by taking the unknowns $x_{c_1}, \cdots, x_{c_r}$ as *dependent* unknowns and using the r equations corresponding to the non–zero rows of B to express these unknowns in terms of the remaining *independent* unknowns $x_{c_{r+1}}, \ldots, x_{c_n}$, which can take on arbitrary values:

$$\begin{aligned} x_{c_1} &= b_{1\,n+1} - b_{1c_{r+1}}x_{c_{r+1}} - \cdots - b_{1c_n}x_{c_n} \\ &\vdots \\ x_{c_r} &= b_{r\,n+1} - b_{rc_{r+1}}x_{c_{r+1}} - \cdots - b_{rc_n}x_{c_n}. \end{aligned}$$

In particular, taking $x_{c_{r+1}} = 0, \ldots, x_{c_{n-1}} = 0$ and $x_{c_n} = 0, 1$ respectively, produces at least two solutions.

EXAMPLE 1.4.1 Solve the system

$$\begin{aligned} x + y &= 0 \\ x - y &= 1 \\ 4x + 2y &= 1. \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

We read off the unique solution $x = \frac{1}{2}$, $y = -\frac{1}{2}$.
(Here $n = 2$, $r = 2$, $c_1 = 1$, $c_2 = 2$. Also $c_r = c_2 = 2 < 3 = n + 1$ and $r = n$.)

EXAMPLE 1.4.2 Solve the system

$$\begin{aligned} 2x_1 + 2x_2 - 2x_3 &= 5 \\ 7x_1 + 7x_2 + x_3 &= 10 \\ 5x_1 + 5x_2 - x_3 &= 5. \end{aligned}$$

Solution. The augmented matrix is

$$A = \begin{bmatrix} 2 & 2 & -2 & 5 \\ 7 & 7 & 1 & 10 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We read off inconsistency for the original system.
(Here $n = 3$, $r = 3$, $c_1 = 1$, $c_2 = 3$. Also $c_r = c_3 = 4 = n + 1$.)

EXAMPLE 1.4.3 Solve the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - x_3 &= 2. \end{aligned}$$

Solution. The augmented matrix is

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}.$$

The complete solution is $x_1 = \frac{3}{2}$, $x_2 = \frac{1}{2} + x_3$, with x_3 arbitrary.
(Here $n = 3$, $r = 2$, $c_1 = 1$, $c_2 = 2$. Also $c_r = c_2 = 2 < 4 = n + 1$ and $r < n$.)

EXAMPLE 1.4.4 Solve the system

$$\begin{aligned} 6x_3 + 2x_4 - 4x_5 - 8x_6 &= 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 &= 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 &= 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 &= 1. \end{aligned}$$

Solution. The augmented matrix is

$$A = \begin{bmatrix} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{bmatrix}$$

which is row equivalent to

$$B = \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{11}{6} & -\frac{19}{6} & 0 & \frac{1}{24} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The complete solution is

$$\begin{aligned} x_1 &= \tfrac{1}{24} + \tfrac{3}{2}x_2 - \tfrac{11}{6}x_4 + \tfrac{19}{6}x_5, \\ x_3 &= \tfrac{5}{3} - \tfrac{1}{3}x_4 + \tfrac{2}{3}x_5, \\ x_6 &= \tfrac{1}{4}, \end{aligned}$$

with x_2, x_4, x_5 arbitrary.
(Here $n = 6$, $r = 3$, $c_1 = 1$, $c_2 = 3$, $c_3 = 6$; $c_r = c_3 = 6 < 7 = n + 1$; $r < n$.)

EXAMPLE 1.4.5 Find the rational number t for which the following system is consistent and solve the system for this value of t.

$$\begin{aligned} x+y &= 2 \\ x-y &= 0 \\ 3x-y &= t. \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 3 & -1 & t \end{bmatrix}$$

which is row–equivalent to the simpler matrix

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & t-2 \end{bmatrix}.$$

Hence if $t \neq 2$ the system is inconsistent. If $t = 2$ the system is consistent and

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We read off the solution $x = 1$, $y = 1$.

EXAMPLE 1.4.6 For which rationals a and b does the following system have (i) no solution, (ii) a unique solution, (iii) infinitely many solutions?

$$\begin{aligned} x-2y+3z &= 4 \\ 2x-3y+az &= 5 \\ 3x-4y+5z &= b. \end{aligned}$$

Solution. The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{bmatrix}$$

$$\left\{\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}\right. \quad \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \quad \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{bmatrix} = B.$$

Case 1. $a \neq 4$. Then $-2a + 8 \neq 0$ and we see that B can be reduced to a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & \frac{b-6}{-2a+8} \end{bmatrix}$$

and we have the unique solution $x = u$, $y = v$, $z = (b-6)/(-2a+8)$.

Case 2. $a = 4$. Then

$$B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & b-6 \end{bmatrix}.$$

If $b \neq 6$ we get no solution, whereas if $b = 6$ then

$$B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + 2R_2 \quad \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ We read off the complete solution $x = -2 + z$, $y = -3 + 2z$, with z arbitrary.

EXAMPLE 1.4.7 Find the reduced row–echelon form of the following matrix over $\mathbb{Z}_3$:

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}.$$

Hence solve the system

$$\begin{aligned} 2x + y + 2z &= 1 \\ 2x + 2y + z &= 0 \end{aligned}$$

over $\mathbb{Z}_3$.

Solution.

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1 \quad \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$R_1 \to 2R_1 \quad \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \quad R_1 \to R_1 + R_2 \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}.$$

The last matrix is in reduced row–echelon form.

To solve the system of equations whose augmented matrix is the given matrix over $\mathbb{Z}_3$, we see from the reduced row–echelon form that $x = 1$ and $y = 2 - 2z = 2 + z$, where $z = 0, 1, 2$. Hence there are three solutions to the given system of linear equations: $(x, y, z) = (1, 2, 0)$, $(1, 0, 1)$ and $(1, 1, 2)$.

1.5 Homogeneous systems

A system of homogeneous linear equations is a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Such a system is always consistent as $x_1 = 0, \cdots, x_n = 0$ is a solution. This solution is called the *trivial* solution. Any other solution is called a *non–trivial* solution.

For example the homogeneous system

$$\begin{aligned} x - y &= 0 \\ x + y &= 0 \end{aligned}$$

has only the trivial solution, whereas the homogeneous system

$$\begin{aligned} x - y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

has the complete solution $x = -z$, $y = 0$, z arbitrary. In particular, taking $z = 1$ gives the non–trivial solution $x = -1$, $y = 0$, $z = 1$.

There is simple but fundamental theorem concerning homogeneous systems.

THEOREM 1.5.1 *A homogeneous system of m linear equations in n unknowns always has a non–trivial solution if $m < n$.*

Proof. Suppose that $m < n$ and that the coefficient matrix of the system is row–equivalent to B, a matrix in reduced row–echelon form. Let r be the number of non–zero rows in B. Then $r \leq m < n$ and hence $n - r > 0$ and so the number $n - r$ of arbitrary unknowns is in fact positive. Taking one of these unknowns to be 1 gives a non–trivial solution.

REMARK 1.5.1 Let two systems of homogeneous equations in n unknowns have coefficient matrices A and B, respectively. If each row of B is a linear combination of the rows of A (i.e. a sum of multiples of the rows of A) and each row of A is a linear combination of the rows of B, then it is easy to prove that the two systems have identical solutions. The converse is true, but is not easy to prove. Similarly if A and B have the same reduced row–echelon form, apart from possibly zero rows, then the two systems have identical solutions and conversely.

There is a similar situation in the case of two systems of linear equations (not necessarily homogeneous), with the proviso that in the statement of the converse, the extra condition that both the systems are consistent, is needed.

1.6 PROBLEMS

1. Which of the following matrices of rationals is in reduced row–echelon form?

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$$

$$\text{(d)} \begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(e)} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(f)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(g)} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ [Answers: (a), (e), (g)]}$$

2. Find reduced row–echelon forms which are row–equivalent to the following matrices:

$$\text{(a)} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}.$$

[Answers:

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.]

3. Solve the following systems of linear equations by reducing the augmented matrix to reduced row–echelon form:

$$(a) \quad \begin{aligned} x+y+z &= 2 \\ 2x+3y-z &= 8 \\ x-y-z &= -8 \end{aligned} \qquad (b) \quad \begin{aligned} x_1+x_2-x_3+2x_4 &= 10 \\ 3x_1-x_2+7x_3+4x_4 &= 1 \\ -5x_1+3x_2-15x_3-6x_4 &= 9 \end{aligned}$$

$$(c) \quad \begin{aligned} 3x-y+7z &= 0 \\ 2x-y+4z &= \tfrac{1}{2} \\ x-y+z &= 1 \\ 6x-4y+10z &= 3 \end{aligned} \qquad (d) \quad \begin{aligned} 2x_2+3x_3-4x_4 &= 1 \\ 2x_3+3x_4 &= 4 \\ 2x_1+2x_2-5x_3+2x_4 &= 4 \\ 2x_1-6x_3+9x_4 &= 7 \end{aligned}$$

[Answers: (a) $x=-3$, $y=\frac{19}{4}$, $z=\frac{1}{4}$; (b) inconsistent;

(c) $x=-\frac{1}{2}-3z$, $y=-\frac{3}{2}-2z$, with z arbitrary;

(d) $x_1=\frac{19}{2}-9x_4$, $x_2=-\frac{5}{2}+\frac{17}{4}x_4$, $x_3=2-\frac{3}{2}x_4$, with x_4 arbitrary.]

4. Show that the following system is consistent if and only if $c=2a-3b$ and solve the system in this case.

$$\begin{aligned} 2x-y+3z &= a \\ 3x+y-5z &= b \\ -5x-5y+21z &= c. \end{aligned}$$

[Answer: $x=\frac{a+b}{5}+\frac{2}{5}z$, $y=\frac{-3a+2b}{5}+\frac{19}{5}z$, with z arbitrary.]

5. Find the value of t for which the following system is consistent and solve the system for this value of t.

$$\begin{aligned} x+y &= 1 \\ tx+y &= t \\ (1+t)x+2y &= 3. \end{aligned}$$

[Answer: $t=2$; $x=1$, $y=0$.]

6. Solve the homogeneous system

$$\begin{aligned} -3x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - 3x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - 3x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 - 3x_4 &= 0. \end{aligned}$$

[Answer: $x_1 = x_2 = x_3 = x_4$, with x_4 arbitrary.]

7. For which rational numbers λ does the homogeneous system

$$\begin{aligned} x + (\lambda - 3)y &= 0 \\ (\lambda - 3)x + y &= 0 \end{aligned}$$

have a non–trivial solution?

[Answer: $\lambda = 2,\ 4$.]

8. Solve the homogeneous system

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0. \end{aligned}$$

[Answer: $x_1 = -\frac{1}{4}x_3$, $x_2 = -\frac{1}{4}x_3 - x_4$, with x_3 and x_4 arbitrary.]

9. Let A be the coefficient matrix of the following homogeneous system of n equations in n unknowns:

$$\begin{aligned} (1-n)x_1 + x_2 + \cdots + x_n &= 0 \\ x_1 + (1-n)x_2 + \cdots + x_n &= 0 \\ \cdots &= 0 \\ x_1 + x_2 + \cdots + (1-n)x_n &= 0. \end{aligned}$$

Find the reduced row–echelon form of A and hence, or otherwise, prove that the solution of the above system is $x_1 = x_2 = \cdots = x_n$, with x_n arbitrary.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix over a field F. Prove that A is row–equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $ad - bc \neq 0$, but is row–equivalent to a matrix whose second row is zero, if $ad - bc = 0$.

11. For which rational numbers a does the following system have (i) no solutions (ii) exactly one solution (iii) infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2. \end{aligned}$$

[Answer: $a = -4$, no solution; $a = 4$, infinitely many solutions; $a \neq \pm 4$, exactly one solution.]

12. Solve the following system of homogeneous equations over $\mathbb{Z}_2$:

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + x_4 + x_5 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ x_3 + x_4 &= 0. \end{aligned}$$

[Answer: $x_1 = x_2 = x_4 + x_5$, $x_3 = x_4$, with x_4 and x_5 arbitrary elements of $\mathbb{Z}_2$.]

13. Solve the following systems of linear equations over $\mathbb{Z}_5$:

$$(a)\quad \begin{aligned} 2x + y + 3z &= 4 \\ 4x + y + 4z &= 1 \\ 3x + y + 2z &= 0 \end{aligned} \qquad (b)\quad \begin{aligned} 2x + y + 3z &= 4 \\ 4x + y + 4z &= 1 \\ x + y &= 3. \end{aligned}$$

[Answer: (a) $x = 1$, $y = 2$, $z = 0$; (b) $x = 1 + 2z$, $y = 2 + 3z$, with z an arbitrary element of $\mathbb{Z}_5$.]

14. If $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are solutions of a system of linear equations, prove that

$$((1 - t)\alpha_1 + t\beta_1, \ldots, (1 - t)\alpha_n + t\beta_n)$$

is also a solution.

15. If $(\alpha_1, \ldots, \alpha_n)$ is a solution of a system of linear equations, prove that the complete solution is given by $x_1 = \alpha_1 + y_1, \ldots, x_n = \alpha_n + y_n$, where $(y_1, \ldots, y_n)$ is the general solution of the associated homogeneous system.

16. Find the values of a and b for which the following system is consistent. Also find the complete solution when $a = b = 2$.

$$\begin{aligned} x + y - z + w &= 1 \\ ax + y + z + w &= b \\ 3x + 2y + \quad aw &= 1 + a. \end{aligned}$$

[Answer: $a \neq 2$ or $a = 2 = b$; $x = 1 - 2z$, $y = 3z - w$, with z, w arbitrary.]

17. Let $F = \{0, 1, a, b\}$ be a field consisting of 4 elements.

(a) Determine the addition and multiplication tables of F. (Hint: Prove that the elements $1+0$, $1+1$, $1+a$, $1+b$ are distinct and deduce that $1+1+1+1 = 0$; then deduce that $1+1 = 0$.)

(b) A matrix A, whose elements belong to F, is defined by

$$A = \begin{bmatrix} 1 & a & b & a \\ a & b & b & 1 \\ 1 & 1 & 1 & a \end{bmatrix},$$

prove that the reduced row–echelon form of A is given by the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Chapter 2

MATRICES

2.1 Matrix arithmetic

A matrix over a field F is a rectangular array of elements from F. The symbol $M_{m\times n}(F)$ denotes the collection of all $m \times n$ matrices over F. Matrices will usually be denoted by capital letters and the equation $A = [a_{ij}]$ means that the element in the i–th row and j–th column of the matrix A equals a_{ij}. It is also occasionally convenient to write $a_{ij} = (A)_{ij}$. For the present, all matrices will have rational entries, unless otherwise stated.

EXAMPLE 2.1.1 The formula $a_{ij} = 1/(i + j)$ for $1 \le i \le 3$, $1 \le j \le 4$ defines a 3×4 matrix $A = [a_{ij}]$, namely

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}.$$

DEFINITION 2.1.1 (Equality of matrices) Matrices A and B are said to be equal if A and B have the same size and corresponding elements are equal; that is A and $B \in M_{m\times n}(F)$ and $A = [a_{ij}]$, $B = [b_{ij}]$, with $a_{ij} = b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$.

DEFINITION 2.1.2 (Addition of matrices) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be of the same size. Then $A + B$ is the matrix obtained by adding corresponding elements of A and B; that is

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

DEFINITION 2.1.3 (Scalar multiple of a matrix) Let $A = [a_{ij}]$ and $t \in F$ (that is t is a *scalar*). Then tA is the matrix obtained by multiplying all elements of A by t; that is

$$tA = t[a_{ij}] = [ta_{ij}].$$

DEFINITION 2.1.4 (Additive inverse of a matrix) Let $A = [a_{ij}]$. Then $-A$ is the matrix obtained by replacing the elements of A by their additive inverses; that is

$$-A = -[a_{ij}] = [-a_{ij}].$$

DEFINITION 2.1.5 (Subtraction of matrices) Matrix subtraction is defined for two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, in the usual way; that is

$$A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}].$$

DEFINITION 2.1.6 (The zero matrix) For each m, n the matrix in $M_{m\times n}(F)$, all of whose elements are zero, is called the *zero* matrix (of size $m \times n$) and is denoted by the symbol 0.

The matrix operations of addition, scalar multiplication, additive inverse and subtraction satisfy the usual laws of arithmetic. (In what follows, s and t will be arbitrary scalars and A, B, C are matrices of the same size.)

1. $(A + B) + C = A + (B + C)$;
2. $A + B = B + A$;
3. $0 + A = A$;
4. $A + (-A) = 0$;
5. $(s + t)A = sA + tA, \quad (s - t)A = sA - tA$;
6. $t(A + B) = tA + tB, \quad t(A - B) = tA - tB$;
7. $s(tA) = (st)A$;
8. $1A = A, \quad 0A = 0, \quad (-1)A = -A$;
9. $tA = 0 \Rightarrow t = 0$ or $A = 0$.

Other similar properties will be used when needed.

DEFINITION 2.1.7 (Matrix product) Let $A = [a_{ij}]$ be a matrix of size $m \times n$ and $B = [b_{jk}]$ be a matrix of size $n \times p$; (that is the number of columns of A equals the number of rows of B). Then AB is the $m \times p$ matrix $C = [c_{ik}]$ whose $(i,\, k)$–th element is defined by the formula

$$c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk} = a_{i1}b_{1k} + \cdots + a_{in}b_{nk}.$$

EXAMPLE 2.1.2

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\times 5 + 2\times 7 & 1\times 6 + 2\times 8 \\ 3\times 5 + 4\times 7 & 3\times 6 + 4\times 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$;

2. $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$;

3. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$;

4. $\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \end{bmatrix}$;

5. $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Matrix multiplication obeys many of the familiar laws of arithmetic apart from the commutative law.

1. $(AB)C = A(BC)$ if A, B, C are $m \times n$, $n \times p$, $p \times q$, respectively;

2. $t(AB) = (tA)B = A(tB), \quad A(-B) = (-A)B = -(AB)$;

3. $(A + B)C = AC + BC$ if A and B are $m \times n$ and C is $n \times p$;

4. $D(A + B) = DA + DB$ if A and B are $m \times n$ and D is $p \times m$.

We prove the associative law only:
First observe that $(AB)C$ and $A(BC)$ are both of size $m \times q$.
Let $A = [a_{ij}]$, $B = [b_{jk}]$, $C = [c_{kl}]$. Then

$$\begin{aligned} ((AB)C)_{il} &= \sum_{k=1}^{p} (AB)_{ik} c_{kl} = \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij}b_{jk} \right) c_{kl} \\ &= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}b_{jk}c_{kl}. \end{aligned}$$

Similarly
$$(A(BC))_{il} = \sum_{j=1}^{n}\sum_{k=1}^{p} a_{ij}b_{jk}c_{kl}.$$
However the double summations are equal. For sums of the form
$$\sum_{j=1}^{n}\sum_{k=1}^{p} d_{jk} \quad \text{and} \quad \sum_{k=1}^{p}\sum_{j=1}^{n} d_{jk}$$
represent the sum of the np elements of the rectangular array $[d_{jk}]$, by rows and by columns, respectively. Consequently
$$((AB)C)_{il} = (A(BC))_{il}$$
for $1 \le i \le m$, $1 \le l \le q$. Hence $(AB)C = A(BC)$.

The system of m linear equations in n unknowns
$$\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\
& \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m
\end{array}$$
is equivalent to a single matrix equation
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$
that is $AX = B$, where $A = [a_{ij}]$ is the *coefficient matrix* of the system, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the *vector of unknowns* and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is the *vector of constants.*

Another useful matrix equation equivalent to the above system of linear equations is
$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

EXAMPLE 2.1.3 The system

$$\begin{aligned} x+y+z &= 1 \\ x-y+z &= 0. \end{aligned}$$

is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and to the equation

$$x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

2.2 Linear transformations

An *n–dimensional column vector* is an $n \times 1$ matrix over F. The collection of all n–dimensional column vectors is denoted by F^n.

Every matrix is associated with an important type of function called a *linear transformation*.

DEFINITION 2.2.1 (Linear transformation) With $A \in M_{m\times n}(F)$, we associate the function $T_A : F^n \to F^m$ defined by $T_A(X) = AX$ for all $X \in F^n$. More explicitly, using components, the above function takes the form

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n, \end{aligned}$$

where y_1, $y_2, \cdots, y_m$ are the components of the column vector $T_A(X)$.

The function just defined has the property that

$$T_A(sX + tY) = sT_A(X) + tT_A(Y) \tag{2.1}$$

for all s, $t \in F$ and all n–dimensional column vectors X, Y. For

$$T_A(sX + tY) = A(sX + tY) = s(AX) + t(AY) = sT_A(X) + tT_A(Y).$$

REMARK 2.2.1 It is easy to prove that if $T : F^n \to F^m$ is a function satisfying equation 2.1, then $T = T_A$, where A is the $m \times n$ matrix whose columns are $T(E_1), \ldots, T(E_n)$, respectively, where $E_1, \ldots, E_n$ are the n–dimensional *unit vectors* defined by

$$E_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad, E_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

One well–known example of a linear transformation arises from rotating the $(x,\, y)$–plane in 2-dimensional Euclidean space, anticlockwise through θ radians. Here a point $(x,\, y)$ will be transformed into the point $(x_1,\, y_1)$, where

$$\begin{aligned} x_1 &= x\cos\theta - y\sin\theta \\ y_1 &= x\sin\theta + y\cos\theta. \end{aligned}$$

In 3–dimensional Euclidean space, the equations

$$x_1 = x\cos\theta - y\sin\theta,\ y_1 = x\sin\theta + y\cos\theta,\ z_1 = z;$$
$$x_1 = x,\ y_1 = y\cos\phi - z\sin\phi,\ z_1 = y\sin\phi + z\cos\phi;$$
$$x_1 = x\cos\psi - z\sin\psi,\ y_1 = y,\ z_1 = x\sin\psi + z\cos\psi;$$

correspond to rotations about the positive z, x, y–axes, anticlockwise through θ, ϕ, ψ radians, respectively.

The product of two matrices is related to the product of the corresponding linear transformations:

If A is $m \times n$ and B is $n \times p$, then the function $T_A T_B : F^p \to F^m$, obtained by first performing T_B, then T_A is in fact equal to the linear transformation T_{AB}. For if $X \in F^p$, we have

$$T_A T_B(X) = A(BX) = (AB)X = T_{AB}(X).$$

The following example is useful for producing rotations in 3–dimensional animated design. (See [27, pages 97–112].)

EXAMPLE 2.2.1 The linear transformation resulting from successively rotating 3–dimensional space about the positive z, x, y–axes, anticlockwise through θ, ϕ, ψ radians respectively, is equal to T_{ABC}, where

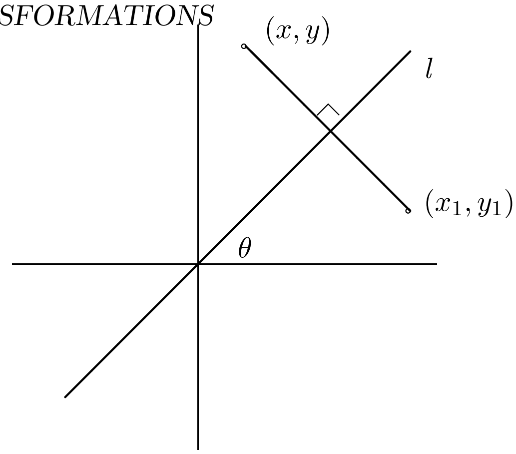


Figure 2.1: Reflection in a line.

$$C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}.$$

$$A = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}.$$

The matrix ABC is quite complicated:

$$A(BC) = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi\cos\theta - \sin\psi\sin\phi\sin\theta & -\cos\psi\sin\theta - \sin\psi\sin\phi\sin\theta & -\sin\psi\cos\phi \\ \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi \\ \sin\psi\cos\theta + \cos\psi\sin\phi\sin\theta & -\sin\psi\sin\theta + \cos\psi\sin\phi\cos\theta & \cos\psi\cos\phi \end{bmatrix}.$$

EXAMPLE 2.2.2 Another example of a linear transformation arising from geometry is reflection of the plane in a line l inclined at an angle θ to the positive x–axis.

We reduce the problem to the simpler case $\theta = 0$, where the equations of transformation are $x_1 = x$, $y_1 = -y$. First rotate the plane clockwise through θ radians, thereby taking l into the x–axis; next reflect the plane in the x–axis; then rotate the plane anticlockwise through θ radians, thereby restoring l to its original position.

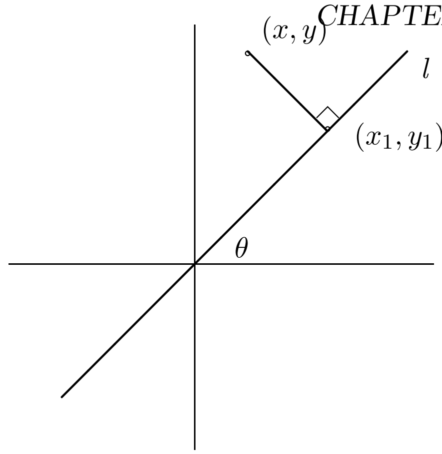


Figure 2.2: Projection on a line.

In terms of matrices, we get transformation equations

$$\begin{aligned}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.\end{aligned}$$

The more general transformation

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = a \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}, \quad a > 0,$$

represents a rotation, followed by a scaling and then by a translation. Such transformations are important in computer graphics. See [23, 24].

EXAMPLE 2.2.3 Our last example of a geometrical linear transformation arises from projecting the plane onto a line l through the origin, inclined at angle θ to the positive x–axis. Again we reduce that problem to the simpler case where l is the x–axis and the equations of transformation are $x_1 = x$, $y_1 = 0$.

In terms of matrices, we get transformation equations

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

2.3 Recurrence relations

DEFINITION 2.3.1 (The identity matrix) The $n \times n$ matrix $I_n = [\delta_{ij}]$, defined by $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, is called the $n \times n$ *identity* matrix of order n. In other words, the columns of the identity matrix of order n are the unit vectors $E_1, \cdots, E_n$, respectively.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

THEOREM 2.3.1 If A is $m \times n$, then $I_m A = A = A I_n$.

DEFINITION 2.3.2 (k–th power of a matrix) If A is an $n \times n$ matrix, we define A^k recursively as follows: $A^0 = I_n$ and $A^{k+1} = A^k A$ for $k \geq 0$.

For example $A^1 = A^0 A = I_n A = A$ and hence $A^2 = A^1 A = AA$.

The usual index laws hold provided $AB = BA$:

1. $A^m A^n = A^{m+n}$, $(A^m)^n = A^{mn}$;
2. $(AB)^n = A^n B^n$;
3. $A^m B^n = B^n A^m$;
4. $(A+B)^2 = A^2 + 2AB + B^2$;
5. $(A+B)^n = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i}$;
6. $(A+B)(A-B) = A^2 - B^2$.

We now state a basic property of the natural numbers.

AXIOM 2.3.1 (PRINCIPLE OF MATHEMATICAL INDUCTION) *If for each $n \geq 1$, $\mathcal{P}_n$ denotes a mathematical statement and*

(i) $\mathcal{P}_1$ *is true,*

(ii) *the truth of* $\mathcal{P}_n$ *implies that of* $\mathcal{P}_{n+1}$ *for each* $n \geq 1$,

then $\mathcal{P}_n$ *is true for all* $n \geq 1$.

EXAMPLE 2.3.1 Let $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$. Prove that

$$A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix} \quad \text{if } n \geq 1.$$

Solution. We use the principle of mathematical induction.

Take $\mathcal{P}_n$ to be the statement

$$A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}.$$

Then $\mathcal{P}_1$ asserts that

$$A^1 = \begin{bmatrix} 1+6\times 1 & 4\times 1 \\ -9\times 1 & 1-6\times 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix},$$

which is true. Now let $n \geq 1$ and assume that $\mathcal{P}_n$ is true. We have to deduce that

$$A^{n+1} = \begin{bmatrix} 1+6(n+1) & 4(n+1) \\ -9(n+1) & 1-6(n+1) \end{bmatrix} = \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -5-6n \end{bmatrix}.$$

Now

$$\begin{aligned} A^{n+1} &= A^nA \\ &= \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix} \\ &= \begin{bmatrix} (1+6n)7+(4n)(-9) & (1+6n)4+(4n)(-5) \\ (-9n)7+(1-6n)(-9) & (-9n)4+(1-6n)(-5) \end{bmatrix} \\ &= \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -5-6n \end{bmatrix}, \end{aligned}$$

and "the induction goes through".

The last example has an application to the solution of a system of *recurrence relations*:

EXAMPLE 2.3.2 The following system of recurrence relations holds for all $n \geq 0$:

$$\begin{aligned} x_{n+1} &= 7x_n + 4y_n \\ y_{n+1} &= -9x_n - 5y_n. \end{aligned}$$

Solve the system for x_n and y_n in terms of x_0 and y_0.

Solution. Combine the above equations into a single matrix equation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

or $X_{n+1} = AX_n$, where $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$ and $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$.
We see that

$$\begin{aligned} X_1 &= AX_0 \\ X_2 &= AX_1 = A(AX_0) = A^2X_0 \\ &\vdots \\ X_n &= A^nX_0. \end{aligned}$$

(The truth of the equation $X_n = A^nX_0$ for $n \geq 1$, strictly speaking follows by mathematical induction; however for simple cases such as the above, it is customary to omit the strict proof and supply instead a few lines of motivation for the inductive statement.)

Hence the previous example gives

$$\begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = X_n &= \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} (1+6n)x_0 + (4n)y_0 \\ (-9n)x_0 + (1-6n)y_0 \end{bmatrix}, \end{aligned}$$

and hence $x_n = (1+6n)x_0 + 4ny_0$ and $y_n = (-9n)x_0 + (1-6n)y_0$, for $n \geq 1$.

2.4 PROBLEMS

1. Let A, B, C, D be matrices defined by

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}.$$

Which of the following matrices are defined? Compute those matrices which are defined.

$$A+B,\ A+C,\ AB,\ BA,\ CD,\ DC,\ D^2.$$

[Answers: $A+C$, BA, CD, D^2;

$$\begin{bmatrix} 0 & -1 \\ 1 & 3 \\ 5 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 12 \\ -4 & 2 \\ -10 & 5 \end{bmatrix}, \quad \begin{bmatrix} -14 & 3 \\ 10 & -2 \\ 22 & -4 \end{bmatrix}, \quad \begin{bmatrix} 14 & -4 \\ 8 & -2 \end{bmatrix}.]$$

2. Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Show that if B is a 3×2 such that $AB = I_2$, then

$$B = \begin{bmatrix} a & b \\ -a-1 & 1-b \\ a+1 & b \end{bmatrix}$$

for suitable numbers a and b. Use the associative law to show that $(BA)^2 B = B$.

3. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, prove that $A^2 - (a+d)A + (ad-bc)I_2 = 0$.

4. If $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$, use the fact $A^2 = 4A - 3I_2$ and mathematical induction, to prove that

$$A^n = \frac{(3^n-1)}{2}A + \frac{3-3^n}{2}I_2 \quad \text{if } n \geq 1.$$

5. A sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ satisfies the recurrence relation $x_{n+1} = ax_n + bx_{n-1}$ for $n \geq 1$, where a and b are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix},$$

where $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and hence express $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in terms of $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$.
If $a = 4$ and $b = -3$, use the previous question to find a formula for x_n in terms of x_1 and x_0.

[Answer:

$$x_n = \frac{3^n - 1}{2} x_1 + \frac{3 - 3^n}{2} x_0.]$$

6. Let $A = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$.

   (a) Prove that

   $$A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix} \quad \text{if } n \geq 1.$$

   (b) A sequence $x_0, x_1, \ldots, x_n, \ldots$ satisfies the recurrence relation $x_{n+1} = 2ax_n - a^2 x_{n-1}$ for $n \geq 1$. Use part (a) and the previous question to prove that $x_n = na^{n-1}x_1 + (1-n)a^n x_0$ for $n \geq 1$.

7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and suppose that λ_1 and λ_2 are the roots of the quadratic polynomial $x^2 - (a+d)x + ad - bc$. (λ_1 and λ_2 may be equal.) Let k_n be defined by $k_0 = 0$, $k_1 = 1$ and for $n \geq 2$

$$k_n = \sum_{i=1}^{n} \lambda_1^{n-i} \lambda_2^{i-1}.$$

Prove that

$$k_{n+1} = (\lambda_1 + \lambda_2)k_n - \lambda_1 \lambda_2 k_{n-1},$$

if $n \geq 1$. Also prove that

$$k_n = \begin{cases} (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2) & \text{if } \lambda_1 \neq \lambda_2, \\ n\lambda_1^{n-1} & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Use mathematical induction to prove that if $n \geq 1$,

$$A^n = k_n A - \lambda_1 \lambda_2 k_{n-1} I_2,$$

[Hint: Use the equation $A^2 = (a+d)A - (ad - bc)I_2$.]

8. Use Question 6 to prove that if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then

$$A^n = \frac{3^n}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{(-1)^{n-1}}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

if $n \geq 1$.

9. The Fibonacci numbers are defined by the equations $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ if $n \geq 1$. Prove that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

if $n \geq 0$.

10. Let $r > 1$ be an integer. Let a and b be arbitrary positive integers. Sequences x_n and y_n of positive integers are defined in terms of a and b by the recurrence relations

$$\begin{aligned} x_{n+1} &= x_n + ry_n \\ y_{n+1} &= x_n + y_n, \end{aligned}$$

for $n \geq 0$, where $x_0 = a$ and $y_0 = b$.

Use Question 6 to prove that

$$\frac{x_n}{y_n} \to \sqrt{r} \quad \text{as } n \to \infty.$$

2.5 Non–singular matrices

DEFINITION 2.5.1 (Non–singular matrix)

A square matrix $A \in M_{n\times n}(F)$ is called *non–singular* or *invertible* if there exists a matrix $B \in M_{n\times n}(F)$ such that

$$AB = I_n = BA.$$

Any matrix B with the above property is called an *inverse* of A. If A does not have an inverse, A is called *singular*.

THEOREM 2.5.1 (Inverses are unique)

If A has inverses B and C, then $B = C$.

Proof. Let B and C be inverses of A. Then $AB = I_n = BA$ and $AC = I_n = CA$. Then $B(AC) = BI_n = B$ and $(BA)C = I_nC = C$. Hence because $B(AC) = (BA)C$, we deduce that $B = C$.

REMARK 2.5.1 If A has an inverse, it is denoted by A^{-1}. So

$$AA^{-1} = I_n = A^{-1}A.$$

Also if A is non–singular, it follows that A^{-1} is also non–singular and

$$(A^{-1})^{-1} = A.$$

THEOREM 2.5.2 If A and B are non–singular matrices of the same size, then so is AB. Moreover

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

Similarly

$$(B^{-1}A^{-1})(AB) = I_n.$$

REMARK 2.5.2 The above result generalizes to a product of m non–singular matrices: If $A_1, \ldots, A_m$ are non–singular $n \times n$ matrices, then the product $A_1 \ldots A_m$ is also non–singular. Moreover

$$(A_1 \ldots A_m)^{-1} = A_m^{-1} \ldots A_1^{-1}.$$

(Thus the inverse of the product equals the product of the inverses *in the reverse order*.)

EXAMPLE 2.5.1 If A and B are $n \times n$ matrices satisfying $A^2 = B^2 = (AB)^2 = I_n$, prove that $AB = BA$.

Solution. Assume $A^2 = B^2 = (AB)^2 = I_n$. Then A, B, AB are non–singular and $A^{-1} = A$, $B^{-1} = B$, $(AB)^{-1} = AB$.

But $(AB)^{-1} = B^{-1}A^{-1}$ and hence $AB = BA$.

EXAMPLE 2.5.2 $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is singular. For suppose $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an inverse of A. Then the equation $AB = I_2$ gives

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and equating the corresponding elements of column 1 of both sides gives the system

$$\begin{aligned} a + 2c &= 1 \\ 4a + 8c &= 0 \end{aligned}$$

which is clearly inconsistent.

THEOREM 2.5.3 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\Delta = ad - bc \neq 0$. Then A is non–singular. Also

$$A^{-1} = \Delta^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

REMARK 2.5.3 The expression $ad - bc$ is called the *determinant* of A and is denoted by the symbols $\det A$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Proof. Verify that the matrix $B = \Delta^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ satisfies the equation $AB = I_2 = BA$.

EXAMPLE 2.5.3 Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{bmatrix}.$$

Verify that $A^3 = 5I_3$, deduce that A is non–singular and find A^{-1}.

Solution. After verifying that $A^3 = 5I_3$, we notice that

$$A\left(\frac{1}{5}A^2\right) = I_3 = \left(\frac{1}{5}A^2\right)A.$$

Hence A is non–singular and $A^{-1} = \frac{1}{5}A^2$.

THEOREM 2.5.4 If the coefficient matrix A of a system of n equations in n unknowns is non–singular, then the system $AX = B$ has the unique solution $X = A^{-1}B$.

Proof. Assume that A^{-1} exists.

1. (Uniqueness.) Assume that $AX = B$. Then

$$\begin{aligned} (A^{-1}A)X &= A^{-1}B, \\ I_nX &= A^{-1}B, \\ X &= A^{-1}B. \end{aligned}$$

2. (Existence.) Let $X = A^{-1}B$. Then

$$AX \;=\; A(A^{-1}B) = (AA^{-1})B = I_nB = B.$$

THEOREM 2.5.5 (Cramer's rule for 2 equations in 2 unknowns)

The system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

has a unique solution if $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, namely

$$x = \frac{\Delta_1}{\Delta}, \qquad y = \frac{\Delta_2}{\Delta},$$

where

$$\Delta_1 = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \quad \text{and} \quad \Delta_2 = \begin{vmatrix} a & e \\ c & f \end{vmatrix}.$$

Proof. Suppose $\Delta \neq 0$. Then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has inverse

$$A^{-1} = \Delta^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and we know that the system

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

has the unique solution

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} e \\ f \end{bmatrix} &= \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} \\ &= \frac{1}{\Delta}\begin{bmatrix} de - bf \\ -ce + af \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} \Delta_1/\Delta \\ \Delta_2/\Delta \end{bmatrix}.\end{aligned}$$

Hence $x = \Delta_1/\Delta$, $y = \Delta_2/\Delta$.

COROLLARY 2.5.1 The homogeneous system

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

has only the trivial solution if $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

EXAMPLE 2.5.4 The system

$$\begin{aligned} 7x + 8y &= 100 \\ 2x - 9y &= 10 \end{aligned}$$

has the unique solution $x = \Delta_1/\Delta$, $y = \Delta_2/\Delta$, where

$$\Delta = \begin{vmatrix} 7 & 8 \\ 2 & -9 \end{vmatrix} = -79,\ \Delta_1 = \begin{vmatrix} 100 & 8 \\ 10 & -9 \end{vmatrix} = -980,\ \Delta_2 = \begin{vmatrix} 7 & 100 \\ 2 & 10 \end{vmatrix} = -130.$$

So $x = \frac{980}{79}$ and $y = \frac{130}{79}$.

THEOREM 2.5.6 Let A be a square matrix. If A is non–singular, the homogeneous system $AX = 0$ has only the trivial solution. Equivalently, if the homogenous system $AX = 0$ has a non–trivial solution, then A is singular.

Proof. If A is non–singular and $AX = 0$, then $X = A^{-1}0 = 0$.

REMARK 2.5.4 If $A_{*1}, \ldots, A_{*n}$ denote the columns of A, then the equation

$$AX = x_1A_{*1} + \ldots + x_nA_{*n}$$

holds. Consequently theorem 2.5.6 tells us that if there exist scalars $x_1, \ldots, x_n$, *not all zero*, such that

$$x_1A_{*1} + \ldots + x_nA_{*n} = 0,$$

that is, if the columns of A are *linearly dependent*, then A is singular. An equivalent way of saying that the columns of A are linearly dependent is that one of the columns of A is expressible as a sum of certain scalar multiples of the remaining columns of A; that is one column is a *linear combination* of the remaining columns.

EXAMPLE 2.5.5

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 4 & 7 \end{bmatrix}$$

is singular. For it can be verified that A has reduced row–echelon form

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and consequently $AX = 0$ has a non–trivial solution $x = -1$, $y = -1$, $z = 1$.

REMARK 2.5.5 More generally, if A is row–equivalent to a matrix containing a zero row, then A is singular. For then the homogeneous system $AX = 0$ has a non–trivial solution.

An important class of non–singular matrices is that of the *elementary row matrices.*

DEFINITION 2.5.2 (Elementary row matrices) There are three types, E_{ij}, $E_i(t)$, $E_{ij}(t)$, corresponding to the three kinds of elementary row operation:

1. E_{ij}, $(i \neq j)$ is obtained from the identity matrix I_n by interchanging rows i and j.

2. $E_i(t)$, $(t \neq 0)$ is obtained by multiplying the i–th row of I_n by t.

3. $E_{ij}(t)$, $(i \neq j)$ is obtained from I_n by adding t times the j–th row of I_n to the i–th row.

EXAMPLE 2.5.6 $(n = 3.)$

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ E_2(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_{23}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The elementary row matrices have the following distinguishing property:

THEOREM 2.5.7 If a matrix A is pre–multiplied by an elementary row–matrix, the resulting matrix is the one obtained by performing the corresponding elementary row–operation on A.

EXAMPLE 2.5.7

$$E_{23}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \\ c & d \end{bmatrix}.$$

COROLLARY 2.5.2 The three types of elementary row–matrices are non–singular. Indeed

1. $E_{ij}^{-1} = E_{ij}$;
2. $E_i^{-1}(t) = E_i(t^{-1})$;
3. $(E_{ij}(t))^{-1} = E_{ij}(-t)$.

Proof. Taking $A = I_n$ in the above theorem, we deduce the following equations:

$$\begin{aligned} E_{ij}E_{ij} &= I_n \\ E_i(t)E_i(t^{-1}) &= I_n = E_i(t^{-1})E_i(t) \quad \text{if } t \neq 0 \\ E_{ij}(t)E_{ij}(-t) &= I_n = E_{ij}(-t)E_{ij}(t). \end{aligned}$$

EXAMPLE 2.5.8 Find the 3×3 matrix $A = E_3(5)E_{23}(2)E_{12}$ explicitly. Also find A^{-1}.

Solution.

$$A = E_3(5)E_{23}(2)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3(5)\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

To find A^{-1}, we have

$$\begin{aligned} A^{-1} &= (E_3(5)E_{23}(2)E_{12})^{-1} \\ &= E_{12}^{-1}\left(E_{23}(2)\right)^{-1}\left(E_3(5)\right)^{-1} \\ &= E_{12}E_{23}(-2)E_3(5^{-1}) \end{aligned}$$

$$\begin{aligned} &= E_{12}E_{23}(-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \\ &= E_{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\frac{2}{5} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}. \end{aligned}$$

REMARK 2.5.6 Recall that A and B are row–equivalent if B is obtained from A by a sequence of elementary row operations. If $E_1, \ldots, E_r$ are the respective corresponding elementary row matrices, then

$$B = E_r\,(\ldots(E_2(E_1A))\ldots) = (E_r \ldots E_1)A = PA,$$

where $P = E_r \ldots E_1$ is non–singular. Conversely if $B = PA$, where P is non–singular, then A is row–equivalent to B. For as we shall now see, P is in fact a product of elementary row matrices.

THEOREM 2.5.8 Let A be non–singular $n \times n$ matrix. Then

(i) A is row–equivalent to I_n,

(ii) A is a product of elementary row matrices.

Proof. Assume that A is non–singular and let B be the reduced row–echelon form of A. Then B has no zero rows, for otherwise the equation $AX = 0$ would have a non–trivial solution. Consequently $B = I_n$.

It follows that there exist elementary row matrices $E_1, \ldots, E_r$ such that $E_r\,(\ldots(E_1A)\ldots) = B = I_n$ and hence $A = E_1^{-1}\ldots E_r^{-1}$, a product of elementary row matrices.

THEOREM 2.5.9 Let A be $n \times n$ and suppose that A is row–equivalent to I_n. Then A is non–singular and A^{-1} can be found by performing the same sequence of elementary row operations on I_n as were used to convert A to I_n.

Proof. Suppose that $E_r \ldots E_1A = I_n$. In other words $BA = I_n$, where $B = E_r \ldots E_1$ is non–singular. Then $B^{-1}(BA) = B^{-1}I_n$ and so $A = B^{-1}$, which is non–singular.

Also $A^{-1} = \left(B^{-1}\right)^{-1} = B = E_r\,((\ldots(E_1I_n)\ldots)$, which shows that A^{-1} is obtained from I_n by performing the same sequence of elementary row operations as were used to convert A to I_n.

REMARK 2.5.7 It follows from theorem 2.5.9 that if A is singular, then A is row–equivalent to a matrix whose last row is zero.

EXAMPLE 2.5.9 Show that $A = \left[\begin{array}{cc} 1 & 2 \\ 1 & 1 \end{array}\right]$ is non–singular, find A^{-1} and express A as a product of elementary row matrices.

Solution. We form the *partitioned* matrix $[A|I_2]$ which consists of A followed by I_2. Then any sequence of elementary row operations which reduces A to I_2 will reduce I_2 to A^{-1}. Here

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{ll}
R_2 \rightarrow R_2 - R_1 & \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \\
R_2 \rightarrow (-1)R_2 & \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \\
R_1 \rightarrow R_1 - 2R_2 & \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right].
\end{array}$$

Hence A is row–equivalent to I_2 and A is non–singular. Also

$$A^{-1} = \left[\begin{array}{cc} -1 & 2 \\ 1 & -1 \end{array}\right].$$

We also observe that

$$E_{12}(-2)E_2(-1)E_{21}(-1)A = I_2.$$

Hence

$$\begin{array}{rcl}
A^{-1} & = & E_{12}(-2)E_2(-1)E_{21}(-1) \\
A & = & E_{21}(1)E_2(-1)E_{12}(2).
\end{array}$$

The next result is the converse of Theorem 2.5.6 and is useful for proving the non–singularity of certain types of matrices.

THEOREM 2.5.10 Let A be an $n \times n$ matrix with the property that the homogeneous system $AX = 0$ has only the trivial solution. Then A is non–singular. Equivalently, if A is singular, then the homogeneous system $AX = 0$ has a non–trivial solution.

Proof. If A is $n \times n$ and the homogeneous system $AX = 0$ has only the trivial solution, then it follows that the reduced row–echelon form B of A cannot have zero rows and must therefore be I_n. Hence A is non–singular.

COROLLARY 2.5.3 Suppose that A and B are $n \times n$ and $AB = I_n$. Then $BA = I_n$.

Proof. Let $AB = I_n$, where A and B are $n \times n$. We first show that B is non–singular. Assume $BX = 0$. Then $A(BX) = A0 = 0$, so $(AB)X = 0$, $I_nX = 0$ and hence $X = 0$.

Then from $AB = I_n$ we deduce $(AB)B^{-1} = I_nB^{-1}$ and hence $A = B^{-1}$. The equation $BB^{-1} = I_n$ then gives $BA = I_n$.

Before we give the next example of the above criterion for non-singularity, we introduce an important matrix operation.

DEFINITION 2.5.3 (The transpose of a matrix) Let A be an $m \times n$ matrix. Then A^t, the *transpose* of A, is the matrix obtained by interchanging the rows and columns of A. In other words if $A = [a_{ij}]$, then $\left(A^t\right)_{ji} = a_{ij}$. Consequently A^t is $n \times m$.

The transpose operation has the following properties:

1. $\left(A^t\right)^t = A$;
2. $(A \pm B)^t = A^t \pm B^t$ if A and B are $m \times n$;
3. $(sA)^t = sA^t$ if s is a scalar;
4. $(AB)^t = B^tA^t$ if A is $m \times n$ and B is $n \times p$;
5. If A is non–singular, then A^t is also non–singular and
$$\left(A^t\right)^{-1} = \left(A^{-1}\right)^t;$$
6. $X^tX = x_1^2 + \ldots + x_n^2$ if $X = [x_1, \ldots, x_n]^t$ is a column vector.

We prove only the fourth property. First check that both $(AB)^t$ and B^tA^t have the same size ($p \times m$). Moreover, corresponding elements of both matrices are equal. For if $A = [a_{ij}]$ and $B = [b_{jk}]$, we have

$$\begin{aligned} \left((AB)^t\right)_{ki} &= (AB)_{ik} \\ &= \sum_{j=1}^{n} a_{ij}b_{jk} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left(B^t\right)_{kj} \left(A^t\right)_{ji} \\
&= \left(B^t A^t\right)_{ki}.
\end{aligned}
$$

There are two important classes of matrices that can be defined concisely in terms of the transpose operation.

DEFINITION 2.5.4 (Symmetric matrix) A real matrix A is called *symmetric* if $A^t = A$. In other words A is square ($n \times n$ say) and $a_{ji} = a_{ij}$ for all $1 \le i \le n$, $1 \le j \le n$. Hence

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

is a general 2×2 symmetric matrix.

DEFINITION 2.5.5 (Skew–symmetric matrix) A real matrix A is called *skew–symmetric* if $A^t = -A$. In other words A is square ($n \times n$ say) and $a_{ji} = -a_{ij}$ for all $1 \le i \le n$, $1 \le j \le n$.

REMARK 2.5.8 Taking $i = j$ in the definition of skew–symmetric matrix gives $a_{ii} = -a_{ii}$ and so $a_{ii} = 0$. Hence

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

is a general 2×2 skew–symmetric matrix.

We can now state a second application of the above criterion for non–singularity.

COROLLARY 2.5.4 Let B be an $n \times n$ skew–symmetric matrix. Then $A = I_n - B$ is non–singular.

Proof. Let $A = I_n - B$, where $B^t = -B$. By Theorem 2.5.10 it suffices to show that $AX = 0$ implies $X = 0$.

We have $(I_n - B)X = 0$, so $X = BX$. Hence $X^tX = X^tBX$.
Taking transposes of both sides gives

$$
\begin{aligned}
(X^tBX)^t &= (X^tX)^t \\
X^tB^t(X^t)^t &= X^t(X^t)^t \\
X^t(-B)X &= X^tX \\
-X^tBX &= X^tX = X^tBX.
\end{aligned}
$$

Hence $X^tX = -X^tX$ and $X^tX = 0$. But if $X = [x_1, \ldots, x_n]^t$, then $X^tX = x_1^2 + \ldots + x_n^2 = 0$ and hence $x_1 = 0, \ldots, x_n = 0$.

2.6 Least squares solution of equations

Suppose $AX = B$ represents a system of linear equations with real coefficients which may be inconsistent, because of the possibility of experimental errors in determining A or B. For example, the system

$$\begin{aligned} x &= 1 \\ y &= 2 \\ x + y &= 3.001 \end{aligned}$$

is inconsistent.

It can be proved that the associated system $A^tAX = A^tB$ is always consistent and that any solution of this system minimizes the sum $r_1^2 + \ldots + r_m^2$, where $r_1, \ldots, r_m$ (the *residuals*) are defined by

$$r_i = a_{i1}x_1 + \ldots + a_{in}x_n - b_i,$$

for $i = 1, \ldots, m$. The equations represented by $A^tAX = A^tB$ are called the *normal equations* corresponding to the system $AX = B$ and any solution of the system of normal equations is called a *least squares* solution of the original system.

EXAMPLE 2.6.1 Find a least squares solution of the above inconsistent system.

Solution. Here $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ 3.001 \end{bmatrix}$.

Then $A^tA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Also $A^tB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3.001 \end{bmatrix} = \begin{bmatrix} 4.001 \\ 5.001 \end{bmatrix}$.

So the normal equations are

$$\begin{aligned} 2x + y &= 4.001 \\ x + 2y &= 5.001 \end{aligned}$$

which have the unique solution

$$x = \frac{3.001}{3}, \quad y = \frac{6.001}{3}.$$

EXAMPLE 2.6.2 Points $(x_1,\, y_1), \ldots, (x_n,\, y_n)$ are experimentally determined and should lie on a line $y = mx + c$. Find a least squares solution to the problem.

Solution. The points have to satisfy

$$\begin{array}{rcl} mx_1 + c & = & y_1 \\ & \vdots & \\ mx_n + c & = & y_n, \end{array}$$

or $Ax = B$, where

$$A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix},\ X = \begin{bmatrix} m \\ c \end{bmatrix},\ B = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

The normal equations are given by $(A^tA)X = A^tB$. Here

$$A^tA = \begin{bmatrix} x_1 & \ldots & x_n \\ 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} = \begin{bmatrix} x_1^2 + \ldots + x_n^2 & x_1 + \ldots + x_n \\ x_1 + \ldots + x_n & n \end{bmatrix}$$

Also

$$A^tB = \begin{bmatrix} x_1 & \ldots & x_n \\ 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 + \ldots + x_ny_n \\ y_1 + \ldots + y_n \end{bmatrix}.$$

It is not difficult to prove that

$$\Delta = \det{(A^tA)} = \sum_{1 \leq i < j \leq n} (x_i - x_j)^2,$$

which is positive unless $x_1 = \ldots = x_n$. Hence if not all of $x_1, \ldots, x_n$ are equal, A^tA is non–singular and the normal equations have a unique solution. This can be shown to be

$$m = \frac{1}{\Delta} \sum_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j),\ c = \frac{1}{\Delta} \sum_{1 \leq i < j \leq n} (x_iy_j - x_jy_i)(x_i - x_j).$$

REMARK 2.6.1 The matrix A^tA is symmetric.

2.7 PROBLEMS

1. Let $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix}$. Prove that A is non–singular, find A^{-1} and express A as a product of elementary row matrices.

   [Answer: $A^{-1} = \begin{bmatrix} \frac{1}{13} & -\frac{4}{13} \\ \frac{3}{13} & \frac{1}{13} \end{bmatrix}$,

   $A = E_{21}(-3)E_2(13)E_{12}(4)$ is one such decomposition.]

2. A square matrix $D = [d_{ij}]$ is called *diagonal* if $d_{ij} = 0$ for $i \neq j$. (That is the *off–diagonal* elements are zero.) Prove that pre–multiplication of a matrix A by a diagonal matrix D results in matrix DA whose rows are the rows of A multiplied by the respective diagonal elements of D. State and prove a similar result for post–multiplication by a diagonal matrix.

   Let $\mathrm{diag}\,(a_1, \ldots, a_n)$ denote the diagonal matrix whose *diagonal* elements d_{ii} are $a_1, \ldots, a_n$, respectively. Show that

   $$\mathrm{diag}\,(a_1, \ldots, a_n)\mathrm{diag}\,(b_1, \ldots, b_n) = \mathrm{diag}\,(a_1b_1, \ldots, a_nb_n)$$

   and deduce that if $a_1 \ldots a_n \neq 0$, then $\mathrm{diag}\,(a_1, \ldots, a_n)$ is non–singular and

   $$(\mathrm{diag}\,(a_1, \ldots, a_n))^{-1} = \mathrm{diag}\,(a_1^{-1}, \ldots, a_n^{-1}).$$

   Also prove that $\mathrm{diag}\,(a_1, \ldots, a_n)$ is singular if $a_i = 0$ for some i.

3. Let $A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 6 \\ 3 & 7 & 9 \end{bmatrix}$. Prove that A is non–singular, find A^{-1} and express A as a product of elementary row matrices.

   [Answers: $A^{-1} = \begin{bmatrix} -12 & 7 & -2 \\ \frac{9}{2} & -3 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$,

   $A = E_{12}E_{31}(3)E_{23}E_3(2)E_{12}(2)E_{13}(24)E_{23}(-9)$ is one such decomposition.]

4. Find the rational number k for which the matrix $A = \begin{bmatrix} 1 & 2 & k \\ 3 & -1 & 1 \\ 5 & 3 & -5 \end{bmatrix}$ is singular. [Answer: $k = -3$.]

5. Prove that $A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is singular and find a non–singular matrix P such that PA has last row zero.

6. If $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix}$, verify that $A^2 - 2A + 13I_2 = 0$ and deduce that $A^{-1} = -\frac{1}{13}(A - 2I_2)$.

7. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

   (i) Verify that $A^3 = 3A^2 - 3A + I_3$.

   (ii) Express A^4 in terms of A^2, A and I_3 and hence calculate A^4 explicitly.

   (iii) Use (i) to prove that A is non–singular and find A^{-1} explicitly.

   [Answers: (ii) $A^4 = 6A^2 - 8A + 3I_3 = \begin{bmatrix} -11 & -8 & -4 \\ 12 & 9 & 4 \\ 20 & 16 & 5 \end{bmatrix}$;

   (iii) $A^{-1} = A^2 - 3A + 3I_3 = \begin{bmatrix} -1 & -3 & 1 \\ 2 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.]

8. (i) Let B be an $n \times n$ matrix such that $B^3 = 0$. If $A = I_n - B$, prove that A is non–singular and $A^{-1} = I_n + B + B^2$.
   Show that the system of linear equations $AX = b$ has the solution

$$X = b + Bb + B^2b.$$

   (ii) If $B = \begin{bmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$, verify that $B^3 = 0$ and use (i) to determine $(I_3 - B)^{-1}$ explicitly.

[Answer: $\begin{bmatrix} 1 & r & s+rt \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$.]

9. Let A be $n \times n$.

   (i) If $A^2 = 0$, prove that A is singular.

   (ii) If $A^2 = A$ and $A \neq I_n$, prove that A is singular.

10. Use Question 7 to solve the system of equations

$$\begin{aligned} x + y - z &= a \\ z &= b \\ 2x + y + 2z &= c \end{aligned}$$

   where a, b, c are given rationals. Check your answer using the Gauss–Jordan algorithm.

   [Answer: $x = -a - 3b + c$, $y = 2a + 4b - c$, $z = b$.]

11. Determine explicitly the following products of 3×3 elementary row matrices.

   (i) $E_{12}E_{23}$ (ii) $E_1(5)E_{12}$ (iii) $E_{12}(3)E_{21}(-3)$ (iv) $(E_1(100))^{-1}$
   (v) E_{12}^{-1} (vi) $(E_{12}(7))^{-1}$ (vii) $(E_{12}(7)E_{31}(1))^{-1}$.

   [Answers: (i) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} -8 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (iv) $\begin{bmatrix} \frac{1}{100} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & -7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (vii) $\begin{bmatrix} 1 & -7 & 0 \\ 0 & 1 & 0 \\ -1 & 7 & 1 \end{bmatrix}$.]

12. Let A be the following product of 4×4 elementary row matrices:

$$A = E_3(2)E_{14}E_{42}(3).$$

   Find A and A^{-1} explicitly.

   [Answers: $A = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}$.]

13. Determine which of the following matrices over $\mathbb{Z}_2$ are non–singular and find the inverse, where possible.

$$\text{(a)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

[Answer: (a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.]

14. Determine which of the following matrices are non–singular and find the inverse, where possible.

$$\text{(a)} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 2 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 4 & 6 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\text{(d)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{(e)} \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \text{(f)} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}.$$

[Answers: (a) $\begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 1 & -1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{2} & 2 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & -1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$.]

15. Let A be a non–singular $n \times n$ matrix. Prove that A^t is non–singular and that $(A^t)^{-1} = (A^{-1})^t$.

16. Prove that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has no inverse if $ad - bc = 0$.

[Hint: Use the equation $A^2 - (a + d)A + (ad - bc)I_2 = 0$.]

17. Prove that the real matrix $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ is non–singular by proving that A is row–equivalent to I_3.

18. If $P^{-1}AP = B$, prove that $P^{-1}A^nP = B^n$ for $n \geq 1$.

19. Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix}$, $P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$. Verify that $P^{-1}AP = \begin{bmatrix} \frac{5}{12} & 0 \\ 0 & 1 \end{bmatrix}$ and deduce that

$$A^n = \frac{1}{7}\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} + \frac{1}{7}\left(\frac{5}{12}\right)^n \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix}.$$

20. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a *Markov* matrix; that is a matrix whose elements are non–negative and satisfy $a+c = 1 = b+d$. Also let $P = \begin{bmatrix} b & 1 \\ c & -1 \end{bmatrix}$. Prove that if $A \neq I_2$ then

    (i) P is non–singular and $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & a+d-1 \end{bmatrix}$,

    (ii) $A^n \to \dfrac{1}{b+c}\begin{bmatrix} b & b \\ c & c \end{bmatrix}$ as $n \to \infty$, if $A \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

21. If $X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$, find XX^t, X^tX, YY^t, Y^tY.

    [Answers: $\begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \\ 17 & 39 & 61 \end{bmatrix}$, $\begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$, $\begin{bmatrix} 1 & -3 & -4 \\ -3 & 9 & 12 \\ -4 & 12 & 16 \end{bmatrix}$, 26.]

22. Prove that the system of linear equations

$$\begin{aligned} x + 2y &= 4 \\ x + y &= 5 \\ 3x + 5y &= 12 \end{aligned}$$

    is inconsistent and find a least squares solution of the system.
    [Answer: $x = 6$, $y = -7/6$.]

23. The points (0, 0), (1, 0), (2, -1), (3, 4), (4, 8) are required to lie on a parabola $y = a + bx + cx^2$. Find a least squares solution for a, b, c. Also prove that no parabola passes through these points.

    [Answer: $a = \frac{1}{5}$, $b = -2$, $c = 1$.]

24. If A is a symmetric $n \times n$ real matrix and B is $n \times m$, prove that $B^t AB$ is a symmetric $m \times m$ matrix.

25. If A is $m \times n$ and B is $n \times m$, prove that AB is singular if $m > n$.

26. Let A and B be $n \times n$. If A or B is singular, prove that AB is also singular.

Chapter 3

SUBSPACES

3.1 Introduction

Throughout this chapter, we will be studying F^n, the set of all n–dimensional column vectors with components from a field F. We continue our study of matrices by considering an important class of subsets of F^n called *subspaces*. These arise naturally for example, when we solve a system of m linear homogeneous equations in n unknowns.

We also study the concept of linear dependence of a family of vectors. This was introduced briefly in Chapter 2, Remark 2.5.4. Other topics discussed are the *row space, column space* and *null space* of a matrix over F, the *dimension* of a subspace, particular examples of the latter being the *rank* and *nullity* of a matrix.

3.2 Subspaces of F^n

DEFINITION 3.2.1 A subset S of F^n is called a subspace of F^n if

1. The zero vector belongs to S; (that is, $0 \in S$);
2. If $u \in S$ and $v \in S$, then $u + v \in S$; (S is said to be closed under vector addition);
3. If $u \in S$ and $t \in F$, then $tu \in S$; (S is said to be closed under scalar multiplication).

EXAMPLE 3.2.1 Let $A \in M_{m\times n}(F)$. Then the set of vectors $X \in F^n$ satisfying $AX = 0$ is a subspace of F^n called the *null space* of A and is denoted here by $N(A)$. (It is sometimes called the *solution space* of A.)

Proof. (1) $A0 = 0$, so $0 \in N(A)$; (2) If $X, Y \in N(A)$, then $AX = 0$ and $AY = 0$, so $A(X+Y) = AX + AY = 0 + 0 = 0$ and so $X + Y \in N(A)$; (3) If $X \in N(A)$ and $t \in F$, then $A(tX) = t(AX) = t0 = 0$, so $tX \in N(A)$.

For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $N(A) = \{0\}$, the set consisting of just the zero vector. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then $N(A)$ is the set of all scalar multiples of $[-2, 1]^t$.

EXAMPLE 3.2.2 Let $X_1, \ldots, X_m \in F^n$. Then the set consisting of all linear combinations $x_1X_1 + \cdots + x_mX_m$, where $x_1, \ldots, x_m \in F$, is a subspace of F^n. This subspace is called the subspace *spanned* or *generated* by $X_1, \ldots, X_m$ and is denoted here by $\langle X_1, \ldots, X_m \rangle$. We also call $X_1, \ldots, X_m$ a spanning family for $S = \langle X_1, \ldots, X_m \rangle$.

Proof. (1) $0 = 0X_1 + \cdots + 0X_m$, so $0 \in \langle X_1, \ldots, X_m \rangle$; (2) If $X, Y \in \langle X_1, \ldots, X_m \rangle$, then $X = x_1X_1 + \cdots + x_mX_m$ and $Y = y_1X_1 + \cdots + y_mX_m$, so

$$\begin{aligned} X + Y &= (x_1X_1 + \cdots + x_mX_m) + (y_1X_1 + \cdots + y_mX_m) \\ &= (x_1 + y_1)X_1 + \cdots + (x_m + y_m)X_m \in \langle X_1, \ldots, X_m \rangle. \end{aligned}$$

(3) If $X \in \langle X_1, \ldots, X_m \rangle$ and $t \in F$, then

$$\begin{aligned} X &= x_1X_1 + \cdots + x_mX_m \\ tX &= t(x_1X_1 + \cdots + x_mX_m) \\ &= (tx_1)X_1 + \cdots + (tx_m)X_m \in \langle X_1, \ldots, X_m \rangle. \end{aligned}$$

For example, if $A \in M_{m\times n}(F)$, the subspace generated by the columns of A is an important subspace of F^m and is called the *column space* of A. The column space of A is denoted here by $C(A)$. Also the subspace generated by the rows of A is a subspace of F^n and is called the *row space* of A and is denoted by $R(A)$.

EXAMPLE 3.2.3 For example $F^n = \langle E_1, \ldots, E_n \rangle$, where $E_1, \ldots, E_n$ are the n–dimensional unit vectors. For if $X = [x_1, \ldots, x_n]^t \in F^n$, then $X = x_1E_1 + \cdots + x_nE_n$.

EXAMPLE 3.2.4 Find a spanning family for the subspace S of $\mathbb{R}^3$ defined by the equation $2x - 3y + 5z = 0$.

Solution. (S is in fact the null space of $[2,\ -3,\ 5]$, so S is indeed a subspace of $\mathbb{R}^3$.)

If $[x,\ y,\ z]^t \in S$, then $x = \frac{3}{2}y - \frac{5}{2}z$. Then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2}y - \frac{5}{2}z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}$$

and conversely. Hence $[\frac{3}{2},\ 1,\ 0]^t$ and $[-\frac{5}{2},\ 0,\ 1]^t$ form a spanning family for S.

The following result is easy to prove:

LEMMA 3.2.1 Suppose each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$. Then any linear combination of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$.

As a corollary we have

THEOREM 3.2.1 Subspaces $\langle X_1, \ldots, X_r\rangle$ and $\langle Y_1, \ldots, Y_s\rangle$ are equal if each of $X_1, \ldots, X_r$ is a linear combination of $Y_1, \ldots, Y_s$ and each of $Y_1, \ldots, Y_s$ is a linear combination of $X_1, \ldots, X_r$.

COROLLARY 3.2.1 Subspaces $\langle X_1, \ldots, X_r,\ Z_1, \ldots, Z_t\rangle$ and $\langle X_1, \ldots, X_r\rangle$ are equal if each of $Z_1, \ldots, Z_t$ is a linear combination of $X_1, \ldots, X_r$.

EXAMPLE 3.2.5 If X and Y are vectors in $\mathbb{R}^n$, then

$$\langle X,\ Y\rangle = \langle X+Y,\ X-Y\rangle.$$

Solution. Each of $X+Y$ and $X-Y$ is a linear combination of X and Y. Also

$$X = \frac{1}{2}(X+Y) + \frac{1}{2}(X-Y) \quad \text{and} \quad Y = \frac{1}{2}(X+Y) - \frac{1}{2}(X-Y),$$

so each of X and Y is a linear combination of $X+Y$ and $X-Y$.

There is an important application of Theorem 3.2.1 to row equivalent matrices (see Definition 1.2.4):

THEOREM 3.2.2 If A is row equivalent to B, then $R(A) = R(B)$.

Proof. Suppose that B is obtained from A by a sequence of elementary row operations. Then it is easy to see that each row of B is a linear combination of the rows of A. But A can be obtained from B by a sequence of elementary operations, so each row of A is a linear combination of the rows of B. Hence by Theorem 3.2.1, $R(A) = R(B)$.

REMARK 3.2.1 If A is row equivalent to B, it is not always true that $C(A) = C(B)$.

For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then B is in fact the reduced row–echelon form of A. However we see that

$$C(A) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$$

and similarly $C(B) = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$.

Consequently $C(A) \neq C(B)$, as $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in C(A)$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin C(B)$.

3.3 Linear dependence

We now recall the definition of linear dependence and independence of a family of vectors in F^n given in Chapter 2.

DEFINITION 3.3.1 Vectors $X_1, \ldots, X_m$ in F^n are said to be *linearly dependent* if there exist scalars $x_1, \ldots, x_m$, *not all zero*, such that

$$x_1 X_1 + \cdots + x_m X_m = 0.$$

In other words, $X_1, \ldots, X_m$ are linearly dependent if some X_i is expressible as a linear combination of the remaining vectors.

$X_1, \ldots, X_m$ are called *linearly independent* if they are not linearly dependent. Hence $X_1, \ldots, X_m$ are linearly independent if and only if the equation

$$x_1 X_1 + \cdots + x_m X_m = 0$$

has only the trivial solution $x_1 = 0, \ldots, x_m = 0$.

EXAMPLE 3.3.1 The following three vectors in $\mathbb{R}^3$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ 7 \\ 12 \end{bmatrix}$$

are linearly dependent, as $2X_1 + 3X_2 + (-1)X_3 = 0$.

REMARK 3.3.1 If $X_1, \ldots, X_m$ are linearly independent and

$$x_1X_1 + \cdots + x_mX_m = y_1X_1 + \cdots + y_mX_m,$$

then $x_1 = y_1, \ldots, x_m = y_m$. For the equation can be rewritten as

$$(x_1 - y_1)X_1 + \cdots + (x_m - y_m)X_m = 0$$

and so $x_1 - y_1 = 0, \ldots, x_m - y_m = 0$.

THEOREM 3.3.1 A family of m vectors in F^n will be linearly dependent if $m > n$. Equivalently, any linearly independent family of m vectors in F^n must satisfy $m \leq n$.

Proof. The equation

$$x_1X_1 + \cdots + x_mX_m = 0$$

is equivalent to n homogeneous equations in m unknowns. By Theorem 1.5.1, such a system has a non–trivial solution if $m > n$.

The following theorem is an important generalization of the last result and is left as an exercise for the interested student:

THEOREM 3.3.2 A family of s vectors in $\langle X_1, \ldots, X_r\rangle$ will be linearly dependent if $s > r$. Equivalently, a linearly independent family of s vectors in $\langle X_1, \ldots, X_r\rangle$ must have $s \leq r$.

Here is a useful criterion for linear independence which is sometimes called the *left–to–right test*:

THEOREM 3.3.3 Vectors $X_1, \ldots, X_m$ in F^n are linearly independent if

(a) $X_1 \neq 0$;

(b) For each k with $1 < k \leq m$, X_k is not a linear combination of $X_1, \ldots, X_{k-1}$.

One application of this criterion is the following result:

THEOREM 3.3.4 Every subspace S of F^n can be represented in the form $S = \langle X_1, \ldots, X_m\rangle$, where $m \leq n$.

Proof. If $S = \{0\}$, there is nothing to prove – we take $X_1 = 0$ and $m = 1$.

So we assume S contains a non–zero vector X_1; then $\langle X_1 \rangle \subseteq S$ as S is a subspace. If $S = \langle X_1 \rangle$, we are finished. If not, S will contain a vector X_2, not a linear combination of X_1; then $\langle X_1, X_2 \rangle \subseteq S$ as S is a subspace. If $S = \langle X_1, X_2 \rangle$, we are finished. If not, S will contain a vector X_3 which is not a linear combination of X_1 and X_2. This process must eventually stop, for at stage k we have constructed a family of k linearly independent vectors $X_1, \ldots, X_k$, all lying in F^n and hence $k \leq n$.

There is an important relationship between the columns of A and B, if A is row–equivalent to B.

THEOREM 3.3.5 Suppose that A is row equivalent to B and let $c_1, \ldots, c_r$ be distinct integers satisfying $1 \leq c_i \leq n$. Then

(a) Columns $A_{*c_1}, \ldots, A_{*c_r}$ of A are linearly dependent if and only if the corresponding columns of B are linearly dependent; indeed more is true:

$$x_1 A_{*c_1} + \cdots + x_r A_{*c_r} = 0 \Leftrightarrow x_1 B_{*c_1} + \cdots + x_r B_{*c_r} = 0.$$

(b) Columns $A_{*c_1}, \ldots, A_{*c_r}$ of A are linearly independent if and only if the corresponding columns of B are linearly independent.

(c) If $1 \leq c_{r+1} \leq n$ and c_{r+1} is distinct from $c_1, \ldots, c_r$, then

$$A_{*c_{r+1}} = z_1 A_{*c_1} + \cdots + z_r A_{*c_r} \Leftrightarrow B_{*c_{r+1}} = z_1 B_{*c_1} + \cdots + z_r B_{*c_r}.$$

Proof. First observe that if $Y = [y_1, \ldots, y_n]^t$ is an n–dimensional column vector and A is $m \times n$, then

$$AY = y_1 A_{*1} + \cdots + y_n A_{*n}.$$

Also $AY = 0 \Leftrightarrow BY = 0$, if B is row equivalent to A. Then (a) follows by taking $y_i = x_{c_j}$ if $i = c_j$ and $y_i = 0$ otherwise.

(b) is logically equivalent to (a), while (c) follows from (a) as

$$\begin{aligned} A_{*c_{r+1}} = z_1 A_{*c_1} + \cdots + z_r A_{*c_r} \quad &\Leftrightarrow \quad z_1 A_{*c_1} + \cdots + z_r A_{*c_r} + (-1) A_{*c_{r+1}} = 0 \\ &\Leftrightarrow \quad z_1 B_{*c_1} + \cdots + z_r B_{*c_r} + (-1) B_{*c_{r+1}} = 0 \\ &\Leftrightarrow \quad B_{*c_{r+1}} = z_1 B_{*c_1} + \cdots + z_r B_{*c_r}. \end{aligned}$$

EXAMPLE 3.3.2 The matrix

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{bmatrix}$$

has reduced row–echelon form equal to

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

We notice that B_{*1}, B_{*2} and B_{*4} are linearly independent and hence so are A_{*1}, A_{*2} and A_{*4}. Also

$$\begin{aligned} B_{*3} &= 2B_{*1} + 3B_{*2} \\ B_{*5} &= (-1)B_{*1} + 2B_{*2} + 3B_{*4}, \end{aligned}$$

so consequently

$$\begin{aligned} A_{*3} &= 2A_{*1} + 3A_{*2} \\ A_{*5} &= (-1)A_{*1} + 2A_{*2} + 3A_{*4}. \end{aligned}$$

3.4 Basis of a subspace

We now come to the important concept of *basis* of a vector subspace.

DEFINITION 3.4.1 Vectors $X_1, \ldots, X_m$ belonging to a subspace S are said to form a basis of S if

(a) Every vector in S is a linear combination of $X_1, \ldots, X_m$;

(b) $X_1, \ldots, X_m$ are linearly independent.

Note that (a) is equivalent to the statement that $S = \langle X_1, \ldots, X_m \rangle$ as we automatically have $\langle X_1, \ldots, X_m \rangle \subseteq S$. Also, in view of Remark 3.3.1 above, (a) and (b) are equivalent to the statement that every vector in S is *uniquely* expressible as a linear combination of $X_1, \ldots, X_m$.

EXAMPLE 3.4.1 The unit vectors $E_1, \ldots, E_n$ form a basis for F^n.

REMARK 3.4.1 The subspace $\{0\}$, consisting of the zero vector alone, does not have a basis. For every vector in a linearly independent family must necessarily be non–zero. (For example, if $X_1 = 0$, then we have the non–trivial linear relation

$$1X_1 + 0X_2 + \cdots + 0X_m = 0$$

and $X_1, \ldots, X_m$ would be linearly dependent.)

However if we exclude this case, every other subspace of F^n has a basis:

THEOREM 3.4.1 A subspace of the form $\langle X_1, \ldots, X_m \rangle$, where at least one of $X_1, \ldots, X_m$ is non–zero, has a basis $X_{c_1}, \ldots, X_{c_r}$, where $1 \leq c_1 < \cdots < c_r \leq m$.

Proof. (The *left–to–right algorithm*). Let c_1 be the least index k for which X_k is non–zero. If $c_1 = m$ or if all the vectors X_k with $k > c_1$ are linear combinations of X_{c_1}, terminate the algorithm and let $r = 1$. Otherwise let c_2 be the least integer $k > c_1$ such that X_k is not a linear combination of X_{c_1}.

If $c_2 = m$ or if all the vectors X_k with $k > c_2$ are linear combinations of X_{c_1} and X_{c_2}, terminate the algorithm and let $r = 2$. Eventually the algorithm will terminate at the r–th stage, either because $c_r = m$, or because all vectors X_k with $k > c_r$ are linear combinations of $X_{c_1}, \ldots, X_{c_r}$.

Then it is clear by the construction of $X_{c_1}, \ldots, X_{c_r}$, using Corollary 3.2.1 that

(a) $\langle X_{c_1}, \ldots, X_{c_r} \rangle = \langle X_1, \ldots, X_m \rangle$;

(b) the vectors $X_{c_1}, \ldots, X_{c_r}$ are linearly independent by the left–to–right test.

Consequently $X_{c_1}, \ldots, X_{c_r}$ form a basis (called the *left–to–right basis*) for the subspace $\langle X_1, \ldots, X_m \rangle$.

EXAMPLE 3.4.2 Let X and Y be linearly independent vectors in $\mathbb{R}^n$. Then the subspace $\langle 0,\ 2X,\ X,\ -Y,\ X+Y \rangle$ has left–to–right basis consisting of $2X$, $-Y$.

A subspace S will in general have more than one basis. For example, any permutation of the vectors in a basis will yield another basis. Given one particular basis, one can determine all bases for S using a simple formula. This is left as one of the problems at the end of this chapter.

We settle for the following important fact about bases:

THEOREM 3.4.2 Any two bases for a subspace S must contain the same number of elements.

Proof. For if $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ are bases for S, then $Y_1, \ldots, Y_s$ form a linearly independent family in $S = \langle X_1, \ldots, X_r \rangle$ and hence $s \leq r$ by Theorem 3.3.2. Similarly $r \leq s$ and hence $r = s$.

DEFINITION 3.4.2 This number is called the *dimension* of S and is written $\dim S$. Naturally we define $\dim \{0\} = 0$.

It follows from Theorem 3.3.1 that for any subspace S of F^n, we must have $\dim S \leq n$.

EXAMPLE 3.4.3 If $E_1, \ldots, E_n$ denote the n–dimensional unit vectors in F^n, then $\dim \langle E_1, \ldots, E_i \rangle = i$ for $1 \leq i \leq n$.

The following result gives a useful way of exhibiting a basis.

THEOREM 3.4.3 A linearly independent family of m vectors in a subspace S, with $\dim S = m$, must be a basis for S.

Proof. Let $X_1, \ldots, X_m$ be a linearly independent family of vectors in a subspace S, where $\dim S = m$. We have to show that every vector $X \in S$ is expressible as a linear combination of $X_1, \ldots, X_m$. We consider the following family of vectors in S: $X_1, \ldots, X_m$, X. This family contains $m+1$ elements and is consequently linearly dependent by Theorem 3.3.2. Hence we have

$$x_1 X_1 + \cdots + x_m X_m + x_{m+1} X = 0, \qquad (3.1)$$

where not all of $x_1, \ldots, x_{m+1}$ are zero. Now if $x_{m+1} = 0$, we would have

$$x_1 X_1 + \cdots + x_m X_m = 0,$$

with not all of $x_1, \ldots, x_m$ zero, contradictiong the assumption that $X_1 \ldots, X_m$ are linearly independent. Hence $x_{m+1} \neq 0$ and we can use equation 3.1 to express X as a linear combination of $X_1, \ldots, X_m$:

$$X = \frac{-x_1}{x_{m+1}} X_1 + \cdots + \frac{-x_m}{x_{m+1}} X_m.$$

3.5 Rank and nullity of a matrix

We can now define three important integers associated with a matrix.

DEFINITION 3.5.1 Let $A \in M_{m\times n}(F)$. Then

(a) column rank $A = \dim C(A)$;

(b) row rank $A = \dim R(A)$;

(c) nullity $A = \dim N(A)$.

We will now see that the reduced row–echelon form B of a matrix A allows us to exhibit bases for the row space, column space and null space of A. Moreover, an examination of the number of elements in each of these bases will immediately result in the following theorem:

THEOREM 3.5.1 Let $A \in M_{m\times n}(F)$. Then

(a) column rank $A =$ row rank A;

(b) column rank $A+$ nullity $A = n$.

Finding a basis for $R(A)$: The r non–zero rows of B form a basis for $R(A)$ and hence row rank $A = r$.

For we have seen earlier that $R(A) = R(B)$. Also

$$\begin{aligned} R(B) &= \langle B_{1*}, \ldots, B_{m*}\rangle \\ &= \langle B_{1*}, \ldots, B_{r*}, 0 \ldots, 0\rangle \\ &= \langle B_{1*}, \ldots, B_{r*}\rangle. \end{aligned}$$

The linear independence of the non–zero rows of B is proved as follows: Let the leading entries of rows $1, \ldots, r$ of B occur in columns $c_1, \ldots, c_r$. Suppose that

$$x_1B_{1*} + \cdots + x_rB_{r*} = 0.$$

Then equating components $c_1, \ldots, c_r$ of both sides of the last equation, gives $x_1 = 0, \ldots, x_r = 0$, in view of the fact that B is in reduced row– echelon form.

Finding a basis for $C(A)$: The r columns $A_{*c_1}, \ldots, A_{*c_r}$ form a basis for $C(A)$ and hence column rank $A = r$. For it is clear that columns $c_1, \ldots, c_r$ of B form the left–to–right basis for $C(B)$ and consequently from parts (b) and (c) of Theorem 3.3.5, it follows that columns $c_1, \ldots, c_r$ of A form the left–to–right basis for $C(A)$.

Finding a basis for $N(A)$: For notational simplicity, let us suppose that $c_1 = 1, \ldots, c_r = r$. Then B has the form

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & b_{1r+1} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{2r+1} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{rr+1} & \cdots & b_{rn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $N(B)$ and hence $N(A)$ are determined by the equations

$$\begin{aligned} x_1 &= (-b_{1r+1})x_{r+1} + \cdots + (-b_{1n})x_n \\ &\vdots \\ x_r &= (-b_{rr+1})x_{r+1} + \cdots + (-b_{rn})x_n, \end{aligned}$$

where $x_{r+1}, \ldots, x_n$ are arbitrary elements of F. Hence the general vector X in $N(A)$ is given by

$$\begin{aligned} \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} &= x_{r+1} \begin{bmatrix} -b_{1r+1} \\ \vdots \\ -b_{rr+1} \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} -b_n \\ \vdots \\ -b_{rn} \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad (3.2) \\ &= x_{r+1}X_1 + \cdots + x_nX_{n-r}. \end{aligned}$$

Hence $N(A)$ is spanned by $X_1, \ldots, X_{n-r}$, as $x_{r+1}, \ldots, x_n$ are arbitrary. Also $X_1, \ldots, X_{n-r}$ are linearly independent. For equating the right hand side of equation 3.2 to 0 and then equating components $r+1, \ldots, n$ of both sides of the resulting equation, gives $x_{r+1} = 0, \ldots, x_n = 0$.

Consequently $X_1, \ldots, X_{n-r}$ form a basis for $N(A)$.

Theorem 3.5.1 now follows. For we have

$$\begin{aligned} \text{row rank } A &= \dim R(A) = r \\ \text{column rank } A &= \dim C(A) = r. \end{aligned}$$

Hence

$$\text{row rank } A = \text{column rank } A.$$

Also

$$\text{column rank}\, A + \text{nullity}\, A = r + \dim N(A) = r + (n - r) = n.$$

DEFINITION 3.5.2 The common value of column rank A and row rank A is called the *rank* of A and is denoted by rank A.

EXAMPLE 3.5.1 Given that the reduced row–echelon form of

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{bmatrix}$$

equal to

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix},$$

find bases for $R(A)$, $C(A)$ and $N(A)$.

Solution. $[1, 0, 2, 0, -1]$, $[0, 1, 3, 0, 2]$ and $[0, 0, 0, 1, 3]$ form a basis for $R(A)$. Also

$$A_{*1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; A_{*2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; A_{*4} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

form a basis for $C(A)$.

Finally $N(A)$ is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 + x_5 \\ -3x_3 - 2x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix} = x_3 X_1 + x_5 X_2,$$

where x_3 and x_5 are arbitrary. Hence X_1 and X_2 form a basis for $N(A)$.

Here rank $A = 3$ and nullity $A = 2$.

EXAMPLE 3.5.2 Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ is the reduced row–echelon form of A.

Hence [1, 2] is a basis for $R(A)$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for $C(A)$. Also $N(A)$ is given by the equation $x_1 = -2x_2$, where x_2 is arbitrary. Then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and hence $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for $N(A)$.

Here rank $A = 1$ and nullity $A = 1$.

EXAMPLE 3.5.3 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the reduced row–echelon form of A.

Hence [1, 0], [0, 1] form a basis for $R(A)$ while [1, 3], [2, 4] form a basis for $C(A)$. Also $N(A) = \{0\}$.

Here rank $A = 2$ and nullity $A = 0$.

We conclude this introduction to vector spaces with a result of great theoretical importance.

THEOREM 3.5.2 Every linearly independent family of vectors in a subspace S can be extended to a basis of S.

Proof. Suppose S has basis $X_1, \ldots, X_m$ and that $Y_1, \ldots, Y_r$ is a linearly independent family of vectors in S. Then

$$S = \langle X_1, \ldots, X_m \rangle = \langle Y_1, \ldots, Y_r,\, X_1, \ldots, X_m \rangle,$$

as each of $Y_1, \ldots, Y_r$ is a linear combination of $X_1, \ldots, X_m$.

Then applying the left–to–right algorithm to the second spanning family for S will yield a basis for S which includes $Y_1, \ldots, Y_r$.

3.6 PROBLEMS

1. Which of the following subsets of $\mathbb{R}^2$ are subspaces?

   (a) $[x,\, y]$ satisfying $x = 2y$;

   (b) $[x,\, y]$ satisfying $x = 2y$ and $2x = y$;

   (c) $[x,\, y]$ satisfying $x = 2y + 1$;

   (d) $[x,\, y]$ satisfying $xy = 0$;

(e) $[x, y]$ satisfying $x \geq 0$ and $y \geq 0$.

[Answer: (a) and (b).]

2. If X, Y, Z are vectors in $\mathbb{R}^n$, prove that

$$\langle X, Y, Z\rangle = \langle X+Y, X+Z, Y+Z\rangle.$$

3. Determine if $X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ are linearly independent in $\mathbb{R}^4$.

4. For which real numbers λ are the following vectors linearly independent in $\mathbb{R}^3$?

$$X_1 = \begin{bmatrix} \lambda \\ -1 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ \lambda \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ \lambda \end{bmatrix}.$$

5. Find bases for the row, column and null spaces of the following matrix over $\mathbb{Q}$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 8 & 11 & 19 & 0 & 11 \end{bmatrix}.$$

6. Find bases for the row, column and null spaces of the following matrix over $\mathbb{Z}_2$:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

7. Find bases for the row, column and null spaces of the following matrix over $\mathbb{Z}_5$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 3 \\ 2 & 1 & 4 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 3 & 2 \end{bmatrix}.$$

8. Find bases for the row, column and null spaces of the matrix A defined in section 1.6, Problem 17. (Note: In this question, F is a field of four elements.)

9. If $X_1, \ldots, X_m$ form a basis for a subspace S, prove that
$$X_1,\ X_1 + X_2, \ldots, X_1 + \cdots + X_m$$
also form a basis for S.

10. Let $A = \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}$. Find conditions on $a,\ b,\ c$ such that (a) $\text{rank}\, A = 1$; (b) $\text{rank}\, A = 2$.

    [Answer: (a) $a = b = c$; (b) at least two of a, b, c are distinct.]

11. Let S be a subspace of F^n with $\dim S = m$. If $X_1, \ldots, X_m$ are vectors in S with the property that $S = \langle X_1, \ldots, X_m \rangle$, prove that $X_1 \ldots, X_m$ form a basis for S.

12. Find a basis for the subspace S of $\mathbb{R}^3$ defined by the equation
$$x + 2y + 3z = 0.$$
Verify that $Y_1 = [-1,\ -1,\ 1]^t \in S$ and find a basis for S which includes Y_1.

13. Let $X_1, \ldots, X_m$ be vectors in F^n. If $X_i = X_j$, where $i < j$, prove that $X_1, \ldots X_m$ are linearly dependent.

14. Let $X_1, \ldots, X_{m+1}$ be vectors in F^n. Prove that
$$\dim \langle X_1, \ldots, X_{m+1} \rangle = \dim \langle X_1, \ldots, X_m \rangle$$
if X_{m+1} is a linear combination of $X_1, \ldots, X_m$, but
$$\dim \langle X_1, \ldots, X_{m+1} \rangle = \dim \langle X_1, \ldots, X_m \rangle + 1$$
if X_{m+1} is not a linear combination of $X_1, \ldots, X_m$.

    Deduce that the system of linear equations $AX = B$ is consistent, if and only if
$$\text{rank}\, [A|B] = \text{rank}\, A.$$

15. Let $a_1, \ldots, a_n$ be elements of F, not all zero. Prove that the set of vectors $[x_1, \ldots, x_n]^t$ where $x_1, \ldots, x_n$ satisfy

$$a_1x_1 + \cdots + a_nx_n = 0$$

is a subspace of F^n with dimension equal to $n - 1$.

16. Prove Lemma 3.2.1, Theorem 3.2.1, Corollary 3.2.1 and Theorem 3.3.2.

17. Let R and S be subspaces of F^n, with $R \subseteq S$. Prove that

$$\dim R \leq \dim S$$

and that equality implies $R = S$. (This is a very useful way of proving equality of subspaces.)

18. Let R and S be subspaces of F^n. If $R \cup S$ is a subspace of F^n, prove that $R \subseteq S$ or $S \subseteq R$.

19. Let $X_1, \ldots, X_r$ be a basis for a subspace S. Prove that all bases for S are given by the family $Y_1, \ldots, Y_r$, where

$$Y_i = \sum_{j=1}^{r} a_{ij} X_j,$$

and where $A = [a_{ij}] \in M_{r \times r}(F)$ is a non–singular matrix.

Similarly

$$(z - e^{\frac{4\pi i}{5}})(z - e^{\frac{-4\pi i}{5}}) = z^2 - 2z\cos\tfrac{4\pi}{5} + 1.$$

This gives the desired factorization.

EXAMPLE 5.7.2 Solve $z^3 = i$.

Solution. $|i| = 1$ and $\operatorname{Arg} i = \frac{\pi}{2} = \alpha$. So by equation 5.4, the solutions are

$$z_k = |i|^{1/3} e^{\frac{i(\alpha+2k\pi)}{3}},\ k = 0,\ 1,\ 2.$$

First, $k = 0$ gives

$$z_0 = e^{\frac{i\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Next, $k = 1$ gives

$$z_1 = e^{\frac{5\pi i}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \frac{-\sqrt{3}}{2} + \frac{i}{2}.$$

Finally, $k = 2$ gives

$$z_1 = e^{\frac{9\pi i}{6}} = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = -i.$$

We finish this chapter with two more examples of De Moivre's theorem.

EXAMPLE 5.7.3 If

$$\begin{aligned} C &= 1 + \cos\theta + \cdots + \cos\,(n-1)\theta, \\ S &= \sin\theta + \cdots + \sin\,(n-1)\theta, \end{aligned}$$

prove that

$$C = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\cos\tfrac{(n-1)\theta}{2} \text{ and } S = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\sin\tfrac{(n-1)\theta}{2},$$

if $\theta \neq 2k\pi,\ k \in \mathbb{Z}$.

Solution.

$$\begin{aligned}
C+iS &= 1+(\cos\theta+i\sin\theta)+\cdots+(\cos\,(n-1)\theta+i\sin\,(n-1)\theta)\\
&= 1+e^{i\theta}+\cdots+e^{i(n-1)\theta}\\
&= 1+z+\cdots+z^{n-1},\ \text{where } z=e^{i\theta}\\
&= \frac{1-z^n}{1-z},\ \text{if } z\neq 1,\ \text{i.e. } \theta\neq 2k\pi,\\
&= \frac{1-e^{in\theta}}{1-e^{i\theta}}=\frac{e^{\frac{in\theta}{2}}(e^{\frac{-in\theta}{2}}-e^{\frac{in\theta}{2}})}{e^{\frac{i\theta}{2}}(e^{\frac{-i\theta}{2}}-e^{\frac{i\theta}{2}})}\\
&= e^{i(n-1)\frac{\theta}{2}}\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\\
&= (\cos\,(n-1)\tfrac{\theta}{2}+i\sin\,(n-1)\tfrac{\theta}{2})\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.
\end{aligned}$$

The result follows by equating real and imaginary parts.

EXAMPLE 5.7.4 Express $\cos\,n\theta$ and $\sin\,n\theta$ in terms of $\cos\theta$ and $\sin\theta$, using the equation $\cos\,n\theta+\sin\,n\theta=(\cos\theta+i\sin\theta)^n$.

Solution. The binomial theorem gives

$$\begin{aligned}(\cos\theta+i\sin\theta)^n &= \cos^n\theta+\tbinom{n}{1}\cos^{n-1}\theta(i\sin\theta)+\tbinom{n}{2}\cos^{n-2}\theta(i\sin\theta)^2+\cdots\\ &\quad+(i\sin\theta)^n.\end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned}
\cos\,n\theta &= \cos^n\theta-\tbinom{n}{2}\cos^{n-2}\theta\sin^2\theta+\cdots\\
\sin\,n\theta &= \tbinom{n}{1}\cos^{n-1}\theta\sin\theta-\tbinom{n}{3}\cos^{n-3}\theta\sin^3\theta+\cdots.
\end{aligned}$$

5.8 PROBLEMS

1. Express the following complex numbers in the form $x+iy$, x,y real:

   (i) $(-3+i)(14-2i)$; (ii) $\dfrac{2+3i}{1-4i}$; (iii) $\dfrac{(1+2i)^2}{1-i}$.

   [Answers: (i) $-40+20i$; (ii) $-\frac{10}{17}+\frac{11}{17}i$; (iii) $-\frac{7}{2}+\frac{i}{2}$.]

2. Solve the following equations:

$$\text{(i)} \qquad iz + (2-10i)z = 3z + 2i,$$

$$\begin{aligned} \text{(ii)} \qquad (1+i)z + (2-i)w &= -3i \\ (1+2i)z + (3+i)w &= 2+2i. \end{aligned}$$

[Answers:(i) $z = -\frac{9}{41} - \frac{i}{41}$; (ii) $z = -1+5i$, $w = \frac{19}{5} - \frac{8i}{5}$.]

3. Express $1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{99}$ in the form $x + iy$, x, y real. [Answer: $(1+2^{50})i$.]

4. Solve the equations: (i) $z^2 = -8-6i$; (ii) $z^2 - (3+i)z + 4 + 3i = 0$.

   [Answers: (i) $z = \pm(1-3i)$; (ii) $z = 2-i$, $1+2i$.]

5. Find the modulus and principal argument of each of the following complex numbers:

   (i) $4+i$; (ii) $-\frac{3}{2} - \frac{i}{2}$; (iii) $-1+2i$; (iv) $\frac{1}{2}(-1+i\sqrt{3})$.

   [Answers: (i) $\sqrt{17}$, $\tan^{-1}\frac{1}{4}$; (ii) $\frac{\sqrt{10}}{2}$, $-\pi + \tan^{-1}\frac{1}{3}$; (iii) $\sqrt{5}$, $\pi - \tan^{-1} 2$.]

6. Express the following complex numbers in modulus-argument form:

   (i) $z = (1+i)(1+i\sqrt{3})(\sqrt{3}-i)$.

   (ii) $z = \dfrac{(1+i)^5(1-i\sqrt{3})^5}{(\sqrt{3}+i)^4}$.

   [Answers:
   (i) $z = 4\sqrt{2}(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$; (ii) $z = 2^{7/2}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$.]

7. (i) If $z = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $w = 3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, find the polar form of
   (a) zw; (b) $\frac{z}{w}$; (c) $\frac{w}{z}$; (d) $\frac{z^5}{w^2}$.

   (ii) Express the following complex numbers in the form $x + iy$:
   (a) $(1+i)^{12}$; (b) $\left(\frac{1-i}{\sqrt{2}}\right)^{-6}$.

   [Answers: (i): (a) $6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$; (b) $\frac{2}{3}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$;

   (c) $\frac{3}{2}(\cos -\frac{\pi}{12} + i\sin -\frac{\pi}{12})$; (d) $\frac{32}{9}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$;

   (ii): (a) -64; (b) $-i$.]

8. Solve the equations:

$$\text{(i) } z^2 = 1 + i\sqrt{3};\ \text{(ii) } z^4 = i;\ \text{(iii) } z^3 = -8i;\ \text{(iv) } z^4 = 2 - 2i.$$

[Answers: (i) $z = \pm\frac{(\sqrt{3}+i)}{\sqrt{2}}$; (ii) $i^k(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}), k = 0, 1, 2, 3$; (iii) $z = 2i,\ -\sqrt{3}-i,\ \sqrt{3}-i$; (iv) $z = i^k 2^{\frac{3}{8}}(\cos\frac{\pi}{16} - i\sin\frac{\pi}{16}),\ k = 0, 1, 2, 3$.]

9. Find the reduced row–echelon form of the complex matrix

$$\begin{bmatrix} 2+i & -1+2i & 2 \\ 1+i & -1+i & 1 \\ 1+2i & -2+i & 1+i \end{bmatrix}.$$

[Answer: $\begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.]

10. (i) Prove that the line equation $lx + my = n$ is equivalent to

$$\overline{p}z + p\overline{z} = 2n,$$

where $p = l + im$.

(ii) Use (ii) to deduce that reflection in the straight line

$$\overline{p}z + p\overline{z} = n$$

is described by the equation

$$\overline{p}w + p\overline{z} = n.$$

[Hint: The complex number $l + im$ is perpendicular to the given line.]

(iii) Prove that the line $|z-a| = |z-b|$ may be written as $\overline{p}z + p\overline{z} = n$, where $p = b - a$ and $n = |b|^2 - |a|^2$. Deduce that if z lies on the Apollonius circle $\frac{|z-a|}{|z-b|} = \lambda$, then w, the reflection of z in the line $|z - a| = |z - b|$, lies on the Apollonius circle $\frac{|z-a|}{|z-b|} = \frac{1}{\lambda}$.

11. Let a and b be distinct complex numbers and $0 < \alpha < \pi$.

(i) Prove that each of the following sets in the complex plane represents a circular arc and sketch the circular arcs on the same diagram:

$$\operatorname{Arg}\frac{z-a}{z-b} = \alpha,\ -\alpha,\ \pi-\alpha,\ \alpha-\pi.$$

Also show that $\operatorname{Arg}\dfrac{z-a}{z-b} = \pi$ represents the line segment joining a and b, while $\operatorname{Arg}\dfrac{z-a}{z-b} = 0$ represents the remaining portion of the line through a and b.

(ii) Use (i) to prove that four distinct points z_1, z_2, z_3, z_4 are concyclic or collinear, if and only if the *cross–ratio*

$$\frac{z_4-z_1}{z_4-z_2}\Big/\frac{z_3-z_1}{z_3-z_2}$$

is real.

(iii) Use (ii) to derive *Ptolemy's* Theorem: Four distinct points A, B, C, D are concyclic or collinear, if and only if one of the following holds:

$$\begin{aligned} AB\cdot CD + BC\cdot AD &= AC\cdot BD \\ BD\cdot AC + AD\cdot BC &= AB\cdot CD \\ BD\cdot AC + AB\cdot CD &= AD\cdot BC. \end{aligned}$$

Chapter 6

EIGENVALUES AND EIGENVECTORS

6.1 Motivation

We motivate the chapter on eigenvalues by discussing the equation

$$ax^2 + 2hxy + by^2 = c,$$

where not all of a, h, b are zero. The expression $ax^2 + 2hxy + by^2$ is called a *quadratic form* in x and y and we have the identity

$$ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = X^t AX,$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$. A is called the matrix of the quadratic form.

We now rotate the x, y axes anticlockwise through θ radians to new x_1, y_1 axes. The equations describing the rotation of axes are derived as follows:

Let P have coordinates (x, y) relative to the x, y axes and coordinates (x_1, y_1) relative to the x_1, y_1 axes. Then referring to Figure 6.1:

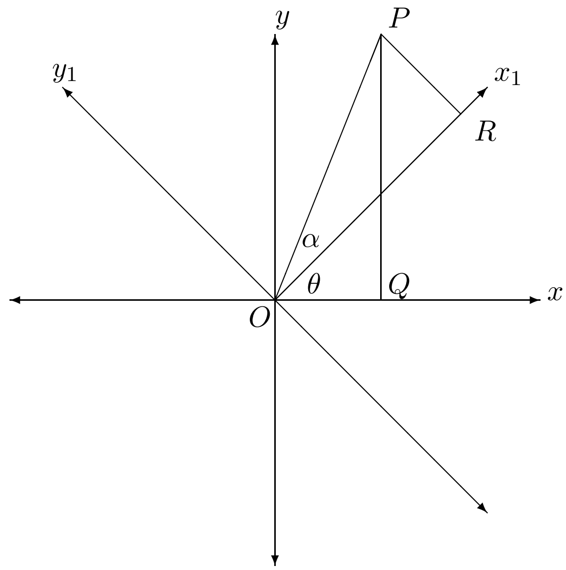


Figure 6.1: Rotating the axes.

$$\begin{aligned}
x &= OQ = OP\cos(\theta+\alpha)\\
&= OP(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\\
&= (OP\cos\alpha)\cos\theta - (OP\sin\alpha)\sin\theta\\
&= OR\cos\theta - PR\sin\theta\\
&= x_1\cos\theta - y_1\sin\theta.
\end{aligned}$$

Similarly $y = x_1\sin\theta + y_1\cos\theta$.

We can combine these transformation equations into the single matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

or $X = PY$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $Y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. We note that the columns of P give the directions of the positive x_1 and y_1 axes. Also P is an orthogonal matrix – we have $PP^t = I_2$ and so $P^{-1} = P^t$. The matrix P has the special property that $\det P = 1$.

A matrix of the type $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is called a *rotation* matrix. We shall show soon that any 2×2 real orthogonal matrix with determinant equal to 1 is a rotation matrix.

We can also solve for the new coordinates in terms of the old ones:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = Y = P^t X = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

so $x_1 = x\cos\theta + y\sin\theta$ and $y_1 = -x\sin\theta + y\cos\theta$. Then

$$X^t AX = (PY)^t A(PY) = Y^t(P^t AP)Y.$$

Now suppose, as we later show, that it is possible to choose an angle θ so that $P^t AP$ is a diagonal matrix, say diag(λ_1, λ_2). Then

$$X^t AX = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 y_1^2 \tag{6.1}$$

and relative to the new axes, the equation $ax^2 + 2hxy + by^2 = c$ becomes $\lambda_1 x_1^2 + \lambda_2 y_1^2 = c$, which is quite easy to sketch. This curve is symmetrical about the x_1 and y_1 axes, with P_1 and P_2, the respective columns of P, giving the directions of the axes of symmetry.

Also it can be verified that P_1 and P_2 satisfy the equations

$$AP_1 = \lambda_1 P_1 \text{ and } AP_2 = \lambda_2 P_2.$$

These equations force a restriction on λ_1 and λ_2. For if $P_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$, the first equation becomes

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \text{ or } \begin{bmatrix} a-\lambda_1 & h \\ h & b-\lambda_1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence we are dealing with a homogeneous system of two linear equations in two unknowns, having a non–trivial solution (u_1, v_1). Hence

$$\begin{vmatrix} a-\lambda_1 & h \\ h & b-\lambda_1 \end{vmatrix} = 0.$$

Similarly, λ_2 satisfies the same equation. In expanded form, λ_1 and λ_2 satisfy

$$\lambda^2 - (a+b)\lambda + ab - h^2 = 0.$$

This equation has real roots

$$\lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4(ab-h^2)}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4h^2}}{2} \tag{6.2}$$

(The roots are distinct if $a \neq b$ or $h \neq 0$. The case $a = b$ and $h = 0$ needs no investigation, as it gives an equation of a circle.)

The equation $\lambda^2 - (a+b)\lambda + ab - h^2 = 0$ is called the *eigenvalue equation* of the matrix A.

6.2 Definitions and examples

DEFINITION 6.2.1 (Eigenvalue, eigenvector)
Let A be a complex square matrix. Then if λ is a complex number and X a *non–zero* complex column vector satisfying $AX = \lambda X$, we call X an *eigenvector* of A, while λ is called an *eigenvalue* of A. We also say that X is an eigenvector corresponding to the eigenvalue λ.

So in the above example P_1 and P_2 are eigenvectors corresponding to λ_1 and λ_2, respectively. We shall give an algorithm which starts from the eigenvalues of $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ and constructs a rotation matrix P such that P^tAP is diagonal.

As noted above, if λ is an eigenvalue of an $n \times n$ matrix A, with corresponding eigenvector X, then $(A - \lambda I_n)X = 0$, with $X \neq 0$, so $\det(A - \lambda I_n) = 0$ and there are at most n distinct eigenvalues of A.

Conversely if $\det(A - \lambda I_n) = 0$, then $(A - \lambda I_n)X = 0$ has a non–trivial solution X and so λ is an eigenvalue of A with X a corresponding eigenvector.

DEFINITION 6.2.2 (Characteristic equation, polynomial)
The equation $\det(A - \lambda I_n) = 0$ is called the *characteristic equation* of A, while the polynomial $\det(A - \lambda I_n)$ is called the *characteristic polynomial* of A. The characteristic polynomial of A is often denoted by $\mathrm{ch}_A(\lambda)$.

Hence the eigenvalues of A are the roots of the characteristic polynomial of A.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, it is easily verified that the characteristic polynomial is $\lambda^2 - (\mathrm{trace}\, A)\lambda + \det A$, where $\mathrm{trace}\, A = a + d$ is the sum of the diagonal elements of A.

EXAMPLE 6.2.1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and find all eigenvectors.

Solution. The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = 0$, or

$$(\lambda - 1)(\lambda - 3) = 0.$$

Hence $\lambda = 1$ or 3. The eigenvector equation $(A - \lambda I_n)X = 0$ reduces to

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{aligned}(2-\lambda)x + y &= 0\\ x + (2-\lambda)y &= 0.\end{aligned}$$

Taking $\lambda = 1$ gives

$$\begin{aligned}x + y &= 0\\ x + y &= 0,\end{aligned}$$

which has solution $x = -y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 1$ are the vectors $\begin{bmatrix} -y \\ y \end{bmatrix}$, with $y \neq 0$.

Taking $\lambda = 3$ gives

$$\begin{aligned}-x + y &= 0\\ x - y &= 0,\end{aligned}$$

which has solution $x = y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 3$ are the vectors $\begin{bmatrix} y \\ y \end{bmatrix}$, with $y \neq 0$.

Our next result has wide applicability:

THEOREM 6.2.1 Let A be a 2×2 matrix having distinct eigenvalues λ_1 and λ_2 and corresponding eigenvectors X_1 and X_2. Let P be the matrix whose columns are X_1 and X_2, respectively. Then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Proof. Suppose $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$. We show that the system of homogeneous equations

$$xX_1 + yX_2 = 0$$

has only the trivial solution. Then by theorem 2.5.10 the matrix $P = [X_1|X_2]$ is non–singular. So assume

$$xX_1 + yX_2 = 0. \tag{6.3}$$

Then $A(xX_1 + yX_2) = A0 = 0$, so $x(AX_1) + y(AX_2) = 0$. Hence

$$x\lambda_1 X_1 + y\lambda_2 X_2 = 0. \tag{6.4}$$

Multiplying equation 6.3 by λ_1 and subtracting from equation 6.4 gives

$$(\lambda_2 - \lambda_1)yX_2 = 0.$$

Hence $y = 0$, as $(\lambda_2 - \lambda_1) \neq 0$ and $X_2 \neq 0$. Then from equation 6.3, $xX_1 = 0$ and hence $x = 0$.

Then the equations $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$ give

$$\begin{aligned} AP = A[X_1|X_2] = [AX_1|AX_2] &= [\lambda_1 X_1|\lambda_2 X_2] \\ &= [X_1|X_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \end{aligned}$$

so

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

EXAMPLE 6.2.2 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ be the matrix of example 6.2.1. Then $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to eigenvalues 1 and 3, respectively. Hence if $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, we have

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

There are two immediate applications of theorem 6.2.1. The first is to the calculation of A^n: If $P^{-1}AP = \mathrm{diag}\,(\lambda_1, \lambda_2)$, then $A = P\mathrm{diag}\,(\lambda_1, \lambda_2)P^{-1}$ and

$$A^n = \left(P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1} \right)^n = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n P^{-1} = P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1}.$$

The second application is to solving a system of linear differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy, \end{aligned}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix of real or complex numbers and x and y are functions of t. The system can be written in matrix form as $\dot{X} = AX$, where

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}.$$

We make the substitution $X = PY$, where $Y = \left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]$. Then x_1 and y_1 are also functions of t and

$$\dot{X} = P\dot{Y} = AX = A(PY), \text{ so } \dot{Y} = (P^{-1}AP)Y = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right] Y.$$

Hence $\dot{x_1} = \lambda_1 x_1$ and $\dot{y_1} = \lambda_2 y_1$.

These differential equations are well–known to have the solutions $x_1 = x_1(0)e^{\lambda_1 t}$ and $x_2 = x_2(0)e^{\lambda_2 t}$, where $x_1(0)$ is the value of x_1 when $t = 0$.

[If $\frac{dx}{dt} = kx$, where k is a constant, then

$$\frac{d}{dt}\left(e^{-kt}x\right) = -ke^{-kt}x + e^{-kt}\frac{dx}{dt} = -ke^{-kt}x + e^{-kt}kx = 0.$$

Hence $e^{-kt}x$ is constant, so $e^{-kt}x = e^{-k0}x(0) = x(0)$. Hence $x = x(0)e^{kt}$.]

However $\left[\begin{array}{c} x_1(0) \\ y_1(0) \end{array}\right] = P^{-1}\left[\begin{array}{c} x(0) \\ y(0) \end{array}\right]$, so this determines $x_1(0)$ and $y_1(0)$ in terms of $x(0)$ and $y(0)$. Hence ultimately x and y are determined as explicit functions of t, using the equation $X = PY$.

EXAMPLE 6.2.3 Let $A = \left[\begin{array}{cc} 2 & -3 \\ 4 & -5 \end{array}\right]$. Use the eigenvalue method to derive an explicit formula for A^n and also solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x - 3y \\ \frac{dy}{dt} &= 4x - 5y, \end{aligned}$$

given $x = 7$ and $y = 13$ when $t = 0$.

Solution. The characteristic polynomial of A is $\lambda^2+3\lambda+2$ which has distinct roots $\lambda_1 = -1$ and $\lambda_2 = -2$. We find corresponding eigenvectors $X_1 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$ and $X_2 = \left[\begin{array}{c} 3 \\ 4 \end{array}\right]$. Hence if $P = \left[\begin{array}{cc} 1 & 3 \\ 1 & 4 \end{array}\right]$, we have $P^{-1}AP = \mathrm{diag}\,(-1, -2)$. Hence

$$\begin{aligned} A^n &= \left(P\mathrm{diag}\,(-1, -2)P^{-1}\right)^n = P\mathrm{diag}\,((-1)^n, (-2)^n)P^{-1} \\ &= \left[\begin{array}{cc} 1 & 3 \\ 1 & 4 \end{array}\right]\left[\begin{array}{cc} (-1)^n & 0 \\ 0 & (-2)^n \end{array}\right]\left[\begin{array}{cc} 4 & -3 \\ -1 & 1 \end{array}\right] \end{aligned}$$

$$
\begin{aligned}
&= (-1)^n \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= (-1)^n \begin{bmatrix} 1 & 3 \times 2^n \\ 1 & 4 \times 2^n \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= (-1)^n \begin{bmatrix} 4 - 3 \times 2^n & -3 + 3 \times 2^n \\ 4 - 4 \times 2^n & -3 + 4 \times 2^n \end{bmatrix}.
\end{aligned}
$$

To solve the differential equation system, make the substitution $X = PY$. Then $x = x_1 + 3y_1$, $y = x_1 + 4y_1$. The system then becomes

$$
\begin{aligned}
\dot{x}_1 &= -x_1 \\
\dot{y}_1 &= -2y_1,
\end{aligned}
$$

so $x_1 = x_1(0)e^{-t}$, $y_1 = y_1(0)e^{-2t}$. Now

$$
\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} -11 \\ 6 \end{bmatrix},
$$

so $x_1 = -11e^{-t}$ and $y_1 = 6e^{-2t}$. Hence $x = -11e^{-t} + 3(6e^{-2t}) = -11e^{-t} + 18e^{-2t}$, $y = -11e^{-t} + 4(6e^{-2t}) = -11e^{-t} + 24e^{-2t}$.

For a more complicated example we solve a system of *inhomogeneous* recurrence relations.

EXAMPLE 6.2.4 Solve the system of recurrence relations

$$
\begin{aligned}
x_{n+1} &= 2x_n - y_n - 1 \\
y_{n+1} &= -x_n + 2y_n + 2,
\end{aligned}
$$

given that $x_0 = 0$ and $y_0 = -1$.

Solution. The system can be written in matrix form as

$$X_{n+1} = AX_n + B,$$

where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

It is then an easy induction to prove that

$$X_n = A^n X_0 + (A^{n-1} + \cdots + A + I_2)B. \tag{6.5}$$

Also it is easy to verify by the eigenvalue method that

$$A^n = \frac{1}{2}\begin{bmatrix} 1+3^n & 1-3^n \\ 1-3^n & 1+3^n \end{bmatrix} = \frac{1}{2}U + \frac{3^n}{2}V,$$

where $U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence

$$\begin{aligned} A^{n-1} + \cdots + A + I_2 &= \frac{n}{2}U + \frac{(3^{n-1} + \cdots + 3 + 1)}{2}V \\ &= \frac{n}{2}U + \frac{(3^{n-1} - 1)}{4}V. \end{aligned}$$

Then equation 6.5 gives

$$X_n = \left(\frac{1}{2}U + \frac{3^n}{2}V\right)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \left(\frac{n}{2}U + \frac{(3^{n-1}-1)}{4}V\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

which simplifies to

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} (2n+1-3^n)/4 \\ (2n-5+3^n)/4 \end{bmatrix}.$$

Hence $x_n = (2n - 1 + 3^n)/4$ and $y_n = (2n - 5 + 3^n)/4$.

REMARK 6.2.1 If $(A - I_2)^{-1}$ existed (that is, if $\det(A - I_2) \neq 0$, or equivalently, if 1 is not an eigenvalue of A), then we could have used the formula

$$A^{n-1} + \cdots + A + I_2 = (A^n - I_2)(A - I_2)^{-1}. \tag{6.6}$$

However the eigenvalues of A are 1 and 3 in the above problem, so formula 6.6 cannot be used there.

Our discussion of eigenvalues and eigenvectors has been limited to 2×2 matrices. The discussion is a more complicated for matrices of size greater than two and is best left to a second course in linear algebra. Nevertheless the following result is a useful generalization of theorem 6.2.1. The reader is referred to [28, page 350] for a proof.

THEOREM 6.2.2 Let A be an $n \times n$ matrix having distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $X_1, \ldots, X_n$. Let P be the matrix whose columns are respectively $X_1, \ldots, X_n$. Then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$